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A Finite Element-Discrete Element Approach for the Analysis of the Venice Trans-Lagoon Railway Bridge

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Abstract

In this paper, the feasibility of the utilization of a combined finite element/discrete element (FE-DE) approach to investigate the behavior of masonry arch bridges is proposed. Attention is paid to the assessment of the load carrying capacity by means of a suitable coupled FE-DE two-dimensional approach. This paper outlines the fields and limits of applicability of the FE-DE method to the study of masonry arch bridges. The main contribution is to evaluate the applicability of FE-DE, in particular its reliability to describe the nonlinear behavior of masonry arch bridges under increasing static loads, to catch kinematic failure mechanisms and collapse load multipliers, as well as to evaluate the role played by the backfill.

A discussion on a possible approach to FE-DE modelling of the Venice trans-Lagoon masonry arch bridge is proposed. With such a purpose, a series of parametric analyses has been conducted in order to evaluate the influence of the different parameters involved on the behavior of the bridges. Pushover analyses have been performed to investigate the nonlinear behavior up to the collapse and up to a clear formation of a failure mechanism in the model.

Keywords: FE-DE, finite element, discrete element, masonry arch bridges, arch-fill interaction, collapse mechanism, pushover analysis, load carrying capacity.

1 Introduction

Thousands of masonry arch bridges belong to different European railways networks. They have been built almost entirely between the second half of the XIX Century and the first half of the XX Century. Considering the high number of masonry arch bridges that are still in service [1], an evaluation of their behavior at collapse may provide interesting information for the assessment of their load bearing capacity in order to ensure their conservation or to design proper intervention of strengthening.

Here an evaluation of the load carrying capacity of masonry arch bridges is proposed. The procedure is based on a combined Finite Element-Discrete Element (FE-DE) 2D approach. In particular the feasibility and reliability of FE-DE approach to investigate the behavior of masonry arch railways bridges is assessed. The numerical model relies into a triangular discretization of the domain with embedded crack elements that activate whenever the peak strength is reached. The proposed approach can be regarded as a combination between Finite Elements allowing for the reproduction of elastic strain into continuum and Discrete Elements, suitable to model frictional cohesive behavior exhibited by masonry structures even at very low levels of external loads. The aforementioned numerical approach is applied to masonry arch bridges interacting with infill.

The analyses have been performed by means of the FE-DE Y2D code [2], in particular using the Y-Geo code developed by the Geo Group of the Toronto University [3] to run the analyses. Analyses are performed under 2D plane stress conditions.

The FE-DE approach has been applied on a valuable case of study, namely the Venice Trans-Lagoon Railway Bridge [4], which is the bridge that connects Venice to its mainland. FE-DE models of the single arch have been made considering or not the presence of backfill, in order to evaluate both the behavior of the masonry arch and the arch-fill interaction. With this purpose, parametric pushover analyses have been performed varying the mechanical properties adopted in the model. A first parametric analysis has been performed on the arch without taking into account the backfill. The arch has been divided in voussoirs separated by joints modeled as elastic-plastic Mohr-Coulomb interfaces. Voussoirs are considered infinite rigid, cracking may occur only in the joints between them. This assumption proved to be suitable to describe the behavior of historical masonry arches.

The parametric analysis has been performed varying the mechanical properties of the joints between the voussoirs, adopting different values of cohesion and friction. The purpose is to assess the value of the collapse load multiplier and the nonlinear behavior of the arch and to evaluate how they change, increasing the mechanical parameters of the joints.

Afterwards, a parametric analysis has been carried out in order to evaluate the backfill role on the global behavior of masonry arch bridges. In backfill, cracking may occur everywhere: crack elements are embedded at the interface of all elements of backfill. The parametric analysis has been performed varying the values of cohesion adopted for backfill.

The pushover analysis has been performed increasing the value of the horizontal load applied until the collapse of the arch. The horizontal load has been considered as an increasing ratio of the vertical load applied, which represents the self-weight of the arch.

2 FE-DE modelling for masonry arch Bridges

Modelling of masonry material is a topical issue. The literature about it is wide. Beside all the different approaches, two main type of models may be distinguished: continuum models based on the Finite Element Method – in which masonry material is modelled as an equivalent continuum obtained by homogenization procedures [5, 6, 7, 8, 9] – and discrete models based on the Discrete Element Method [10, 11, 12, 13, 14].

In recent times an increasing number of models attempted to combine the advantage of Finite and Discrete Element methods. One of these approaches is the combined Finite-Discrete Element method (FE-DE) proposed by Munjiza [15, 2]. FE-DE Y2D code has been recently updated by the Geo Group of the Toronto University [16, 17]. In the last years, FE-DE method has been successfully adopted to study the behavior of historical masonry structures [18, 19, 20, 21].

Many authors dealt with structural behavior of masonry arch. Besides the historic rules [22], the classic approach to determine the stability of arch bridges is probably due to Pippard and Ashby [23, 24] and Heyman [25]. Heyman [25] was the first to extend in a clear and explicit way to masonry arches both the kinematic and static theorems of limit analysis, according to which the structure is safe if a thrust line inner to the arch depth can be determined in equilibrium with the external loads.

As previously stated, the most common idealizations of masonry material behavior are elastic, nonlinear elastic and elastic plastic [9], but in the case of masonry arch bridges and curved structures in general the most diffused approach still remains limit analysis [26]. Several rigid blocks analysis methods have been developed to study the behavior at collapse of masonry arch [27, 28, 29]. This approach is based on a rigid block discretization of the arches within limit analysis concepts coupled with Finite Elements. While such an approach is very appealing because it provides failure mechanisms and load multipliers for a variety of different 2D geometries and loading conditions, still it is based on strong simplifications and consider the role played by the backfill only in an approximate way.

To rigorously investigate the role played by the backfill in the determination of the actual load carrying capacity of 2D bridges, a discretization with plane strain rigid-plastic elements and interfaces is needed, as recently proposed by Cavicchi and Gambarotta [30, 31, 32]. The role of backfill respect to both service and ultimate loads and the transversal effect of load have been studied by the authors [4].

Discrete models and FE-DE model have been proposed for stone arch and masonry arch bridges [33, 34, 35].

The approach to FE-DE analysis of masonry arch bridges is here presented. The numerical model relies into a triangular discretization of the domain with embedded crack elements that activate whenever the peak strength is reached.

The analyses have been performed by means of the FE-DE Y2D code [2], in particular using the Y-GUI [16] to prepare the input file and the Y-Geo code developed by the Geo Group of the Toronto University [17] to run the analyses. Analyses are performed under 2D plane stress conditions.

Bridges have been modelled adopting Finite Element meshes. The properties adopted for Finite Elements are Young's modulus E^B and Poisson's coefficient ν, density ρ and viscous damping μ. In order to avoid compenetration of blocks, a penalty contact parameter is adopted, equal to the Young's modulus E^B, and a tangential penalty adopted, equal to its half.

A series of parametric analyses have been performed to evaluate the influence of the different mechanical parameters adopted for the joints respect to the behavior of the bridge. Joints are modelled as elastic-plastic Mohr-Coulomb interface. Parametric analyses have been carried out varying the parameters involved: cohesion c and friction φ, and then the other related parameters, tensile strength τ and fracture energy GI_C (first mode) and GII_C (second mode) which have been calculated on the base of the values of cohesion and friction adopted.

A first parametric analysis has been performed on the arch without taking into account the backfill. The arch has been divided in wedges, called voussoirs, separated by joints modeled as elastic-plastic Mohr-Coulomb interface. Voussoirs are considered infinite rigid, by adopting a very high value of Young's Modulus E^B, a Poisson's coefficient v equal to zero and a very strong internal joints, in order to avoid cracking inside the voussoirs, that may occur only in the joints between them. This assumption is suitable to describe the behavior of historical masonry arch.

The parametric analysis has been performed varying the mechanical properties of the joints between the voussoirs; therefore different values of cohesion c have been adopted. The purpose is to assess the value of the collapse load multiplier and the nonlinear behavior of the arch and to evaluate how they change at the increasing of the mechanical parameters of the joints. Therefore, a pushover analysis has been performed for each value of cohesion and friction adopted and then the results of the different analyses have been compared.

The pushover analysis has been performed increasing the value of the horizontal load applied until the collapse of the arch. The horizontal load has been considered as an increasing ratio of the vertical load applied, which represents the self-weight of the arch. Therefore, several incremental nonlinear dynamic analyses have been performed for each increasing value of horizontal loads.

Afterwards, a parametric analysis has been carried out in order to evaluate the backfill role on the global behavior of masonry arch bridges. In backfill, cracking may occur everywhere: crack elements are embedded at the interface of all elements of backfill and only one set of mechanical properties for the backfill joint is considered (differently from the arch, in which internal joints of voussoirs and joints between voussoirs are modelled with different mechanical properties, in order to allow cracking only between voussoirs). The position of crack depends on the mesh: fine meshes should be preferred, but taking into account also the computational costs needed. A very high value of friction angle has been adopted for backfill, $\varphi = 50°$: the adoption of a high angle of friction provides a value of tensile strength lower than cohesion, which seems to be appropriate to simulate the behavior of incoherent filling. The properties of backfill internal joints have been adopted also to model arch-fill interaction.

Parametric analyses have been performed on a study case the Venice trans-Lagoon Bridge.

3 Venice trans-Lagoon masonry arch bridge

The Venice trans-Lagoon bridge is the bridge that connects Venice to its mainland, in Italy. The bridge was built in 1846 and during its life was subjected to several interventions and enlargements. At the moment it consists of three different bridges coupled: the historical masonry arch bridge, the roadway bridge built in 1933 and the new rail bridge, built in 1973. The historical bridge is hidden by the new bridges that have been built, and it is partially connected to them. However here only the historical masonry arch bridge is considered. An image of the historical bridge is shown in Figure 1. Previous studies have been conducted on the bridge by means of continuous FEM analysis with 3D homogenization procedure [36, 4,], here 2D FE-DE analyses are performed. The bridge carries two rails belonging to the railway Milan-Venice, which is a main railroad in Italy, highly congested by traffic.

Figure 1: The Venice trans-Lagoon Bridge, Italy, during the XIX century.

The bridge has a total length of 3600 m, it consists of 222 arches divided in 6 modules of 37 arches each one, named *stadii*, which are separated by artificial islands. Each *stadio* is divided in 7 sequences of 5 arches, except the central one consisting of 7 arches: between each sequence there is a big pier in order to prevent a global collapse due to the fall down of a single arch. For this reason the bridge could be considered as a sum of minor bridges made of 5 or 7 arches. Each arch has a span of 10 m and a rise of 1.73 m, with a ratio span/rise $R_{s/r}$ equal to about 1:5.8. The vault, made by bricks and mortar joints, has a curvature radius of 8.80 m at intrados and a transversal depth of 9 m. The thickness changes: 0.65 m at the crown, 0.80 m in the half of middle span, 0.94 at the abutments. However here its thickness is considered constant equal to 0.80 m, the ratio thickness/span is $R_{t/s}$ = 1/12.5. The barrel vault is completely made by bricks and mortar joints. Abutments and piers are made of Istrian stones. Backfill is made by heterogeneous incoherent materials.

Here analysis is performed on a single arch. Only the arch and backfill systems have been analyzed, without modelling piers.

The first parametric analysis has been performed on the arch without backfill. As previously stated, the arch has been divided in wedges separated by joints modeled as elastic-plastic Mohr-Coulomb interfaces. Blocks are considered infinite rigid and cracks may occur only in the joints between the blocks. A parametric analysis has been performed varying cohesion: values $c = 0.10$ MPa, $c = 0.20$ MPa and $c = 0.25$ MPa have been adopted, while friction has been kept constant $\varphi = 37°$.

A pushover analysis has been performed by increasing the value of horizontal load up to the collapse of the bridge. Vertical and horizontal loads are applied at the center of blocks. A depth of 1 m is considered. The volume of blocks V^B is (0.80 x 0.55 x 1) = 0.46 m^3 and the density is $\rho = 20$ kN/m^3, therefore the vertical load applied in each block is $P^B = 8.556$ kN. The horizontal load applied is equal to $H^B = \lambda P^B$, where λ increases up to the value needed to activate the mechanism of collapse of the arch. As before, two additional blocks are added at the abutments to allow possible cracking.

The geometry of the bridge, considering only arch and backfill, is shown in Figure 2, the FE-DE model of the Venice trans-Lagoon Bridge arch and the loads applied are reported in Figure 3.

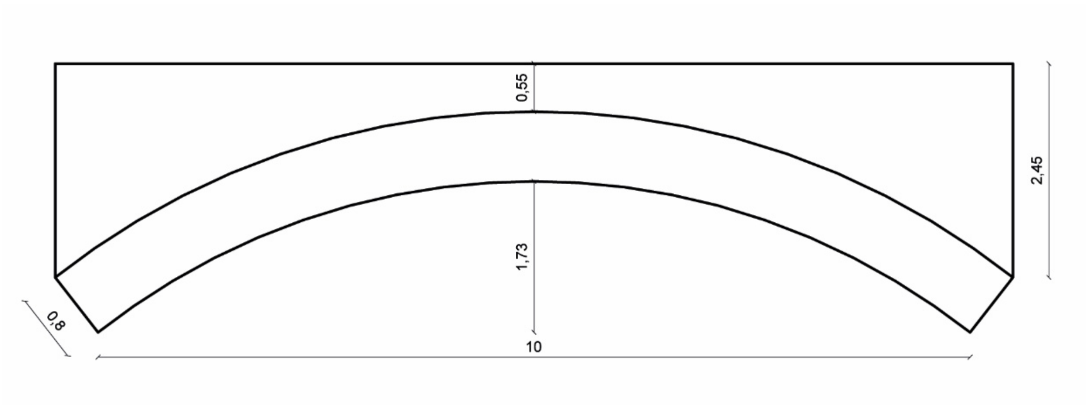


Figure 2: Geometry of the Venice trans-Lagoon Bridge, dimension in meters.

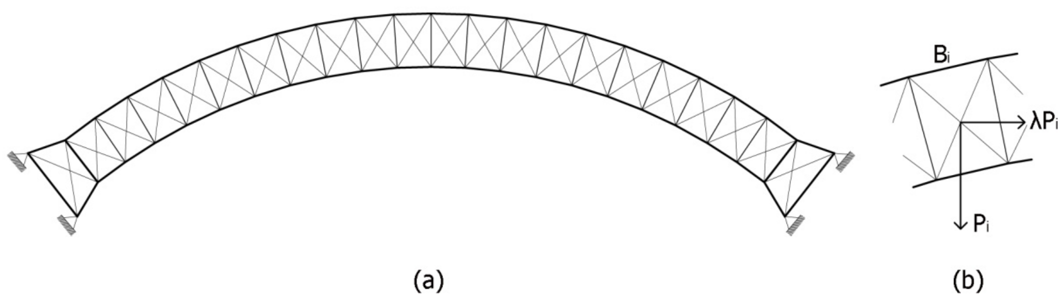


Figure 3: FE-DE model of the Venice trans-Lagoon Bridge arch (a) and loads applied (b)

The mechanical properties of blocks and the parameters of the joints between them are reported in Tables 1 and 2. In order to simulate rigid blocks, a very high value of Young's modulus E^B has been adopted with Poisson's ratio $\nu = 0$. Internal joints inside blocks are modeled to avoid cracking, while the joints between blocks are modeled varying cohesion.

Young Modulus E^B (MPa)	Poisson Ratio v	Viscous Damping μ	Density ρ (kN/m^3)	Contact Penalty (MPa)	Tangential Penalty (MPa)
2E+06	0	3.16E+04	20	2E+06	1E+05

Table 1: Mechanical properties of blocks.

Cohesion c (MPa)	Friction φ (°)	Tensile strength τ (MPa)	Fracture Energy I GI_C	Fracture Energy II GII_C
0.10		0.13	2.5E-05	4.4E-05
0.20	37	0.26	1.0E-04	1.77E-04
0.25		0.33	1.57E-04	2.77E-04

Table 2: Mechanical properties of joints.

Results of the parametric analysis are reported in figure 4. It is possible to notice that, increasing the value of cohesion, an increase of the collapse load multiplier is obtained, obviously in agreement with intuition. However also the collapse behavior changes: for low value of cohesion c = 0.10 MPa the collapse occurs with the typical 4 hinges mechanism, while with higher values of cohesion c = 0.20 or 0.25 MPa, the collapse occurs more swiftly because one of the joints cracks by slippage, as shown in figure 5.

In figure 4 it is possible to notice that at the beginning the arch is lowered due to its self-weight, thus vertical displacements are negative, while once the mechanism is activated vertical displacements become positive, because the crown undergoes a rise due to the four hinges mechanisms.

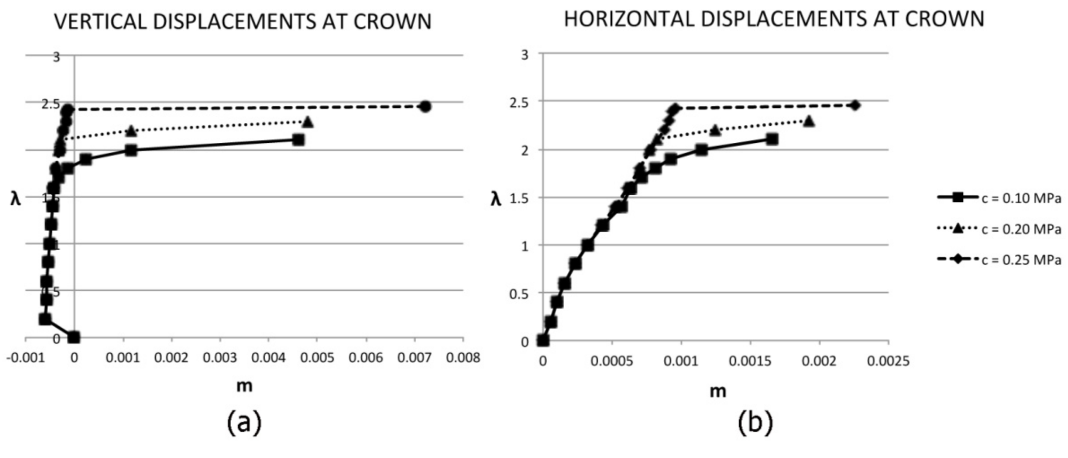
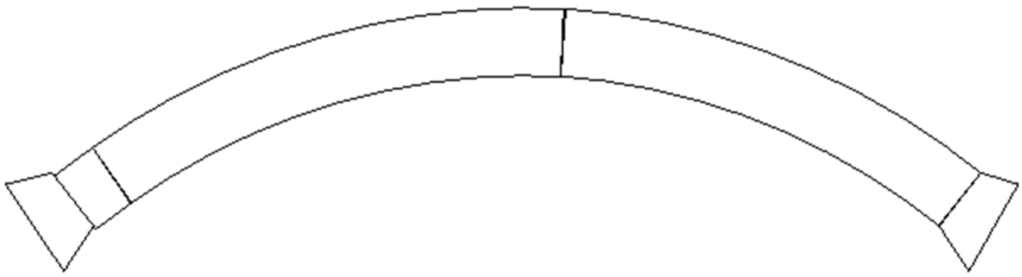
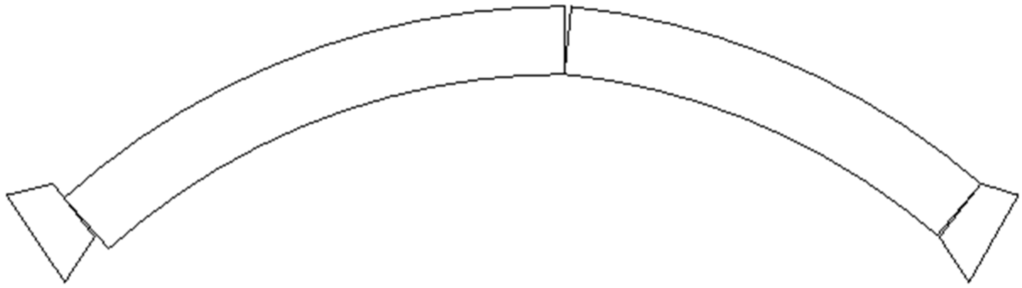


Figure 4: Vertical (a) and horizontal (b) displacements at crown for different values of cohesion c.



(a)



(b)

Figure 5: Different collapse mechanism of Venice trans-Lagoon Bridge arch, (a) cohesion c = 0.10 MPa, (b) cohesion c = 0.25 MPa.

Afterwards to the first parametric analysis, backfill effect has been taken into account. A new mesh has been prepared starting from the first one: triangular elements representing the backfill profile have been added to the original arch. The mechanical properties adopted for the backfill are reported in Table 3, while the same properties have been kept for the arch and its joints. Backfill has not been considered rigid, so its Young's modulus E^F is considerably lower respect to the arch, E^F = 1400 MPa. A parametric analysis has been performed varying the cohesion of the backfill: c = 0.04 MPa and c = 0.08 MPa have been adopted, while friction angle has been kept constant φ =50°. The mechanical properties of backfill internal joints are summarized in Table 4.

Young Modulus E^F (MPa)	**Poisson Coefficient ν**	**Viscous Damping μ**	**Density ρ (kN/m^3)**	**Contact Penalty (MPa)**	**Tangential Penalty (MPa)**
1.4E+3	0	1.34E+02	20	1.4E+03	7.0E+02

Table 3: Mechanical properties of backfill

Cohesion c (MPa)	**Friction φ (°)**	**Tensile strength τ (MPa)**	**Fracture Energy I GI_C**	**Fracture Energy II GII_C**
0.04	50	0.033	6.5E-07	4.6E-07
0.08		0.067	2.59E-06	1.81E-06

Table 4: Mechanical properties of internal backfill joints

Vertical and horizontal loads of backfill are distributed along the mesh, on the nodes belonging to each column of backfill over the arch. Boundary conditions applied are the same adopted for the abutments: they are fixed, while the external sides of backfill have horizontal restraints, in order to avoid contraction. The FE-DE model and the portions of backfill considered for self-weight are reported in Figure 6.

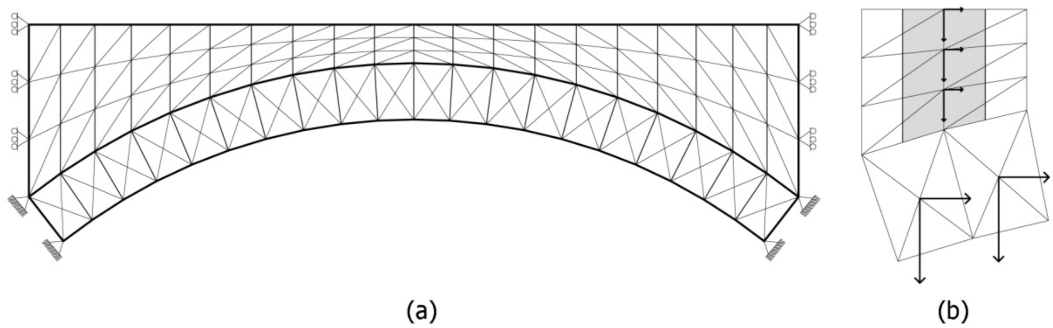


Figure 6: FE-DE model of the Venice trans-Lagoon Bridge (a) and loads applied (b)

Results obtained confirm the capacity of FE-DE to take into account the effect of backfill to the global behavior. Graphs reported in figure 7 show a comparison between the behavior of the single arch and of the arch with the backfill. In particular it is possible to notice that the presence of backfill delays the begin of the nonlinear behavior and increases the collapse load multiplier.

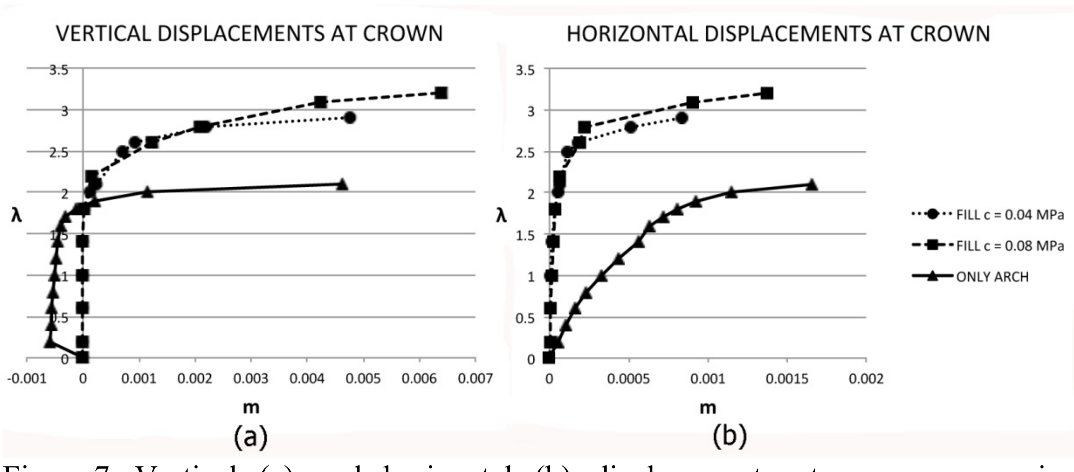
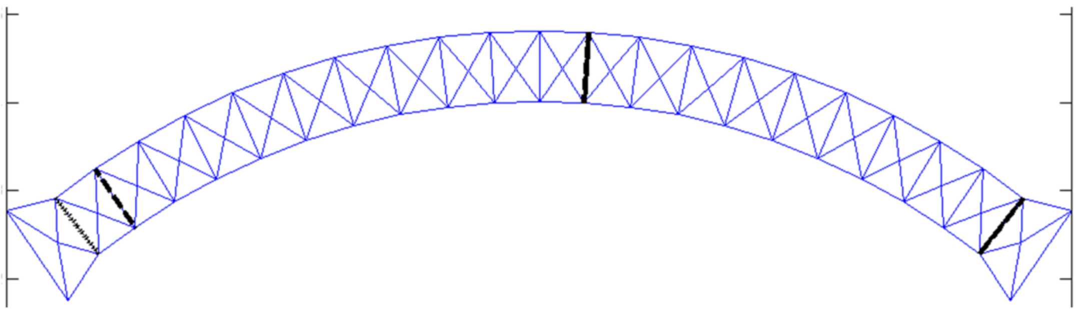


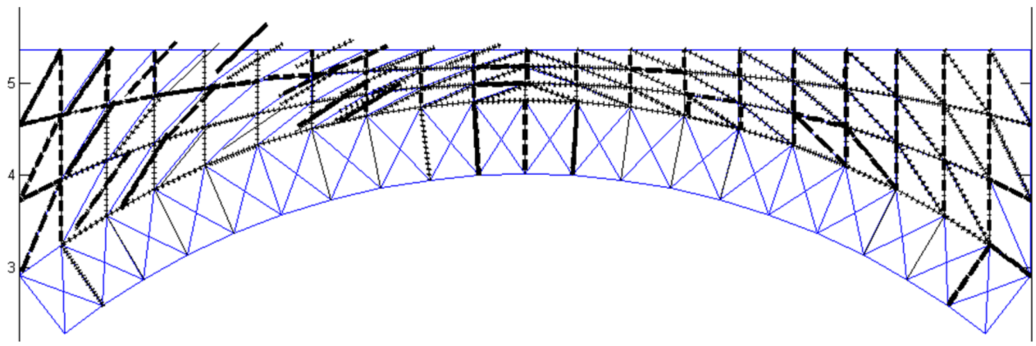
Figure 7: Vertical (a) and horizontal (b) displacements at crown comparison between single arch and arch + backfill

The presence of backfill reduces the horizontal displacements of the arch, as highlighted in the graph that plot horizontal displacements at crown. Moreover the stabilizing effect of backfill is clearly shown in the graph that plots vertical displacements at crown. In the case of the single arch, at the beginning the arch exhibits negative vertical displacements that became positive when the kinematic mechanism is activated. The presence of backfill reduces the initial negative vertical displacement of at least one order of magnitude and the kinematic mechanism is activated for higher values of load.

The backfill cracks before the arch, which collapses only when the stabilizing effect of the backfill ends. Moreover, crack elements that break in the model of both the single arch and the whole bridge (arch+backfill) are reported in Figure 8. The picture is obtained by means of an ad hoc MatLab script. Line types describe the type of failure: the mesh is in solid light line; crack elements broken represented by dashed line are subjected to de-cohesion, by dotted line to sliding and by dash-dot line to a mixed failure mode. It is possible to notice how the mechanism changes considerably due to the backfill presence.



(a)



(b)

Figure 7: Crack elements broken, (a) single arch c = 0.10 MPa, (b) backfill c = 0.08 MPa (dashed line: de-cohesion; dotted line: sliding; dash-dot line: mixed mode)

4 Conclusion

FE-DE modelling seems a to be consistent procedure to describe the nonlinear behavior of masonry arch bridges. It allows to take into account both the characteristics of masonry material and the phenomena characterizing the behavior of a masonry arch. It allows modelling the masonry arch as made by rigid blocks, the voissoirs, that may activate a kinematic mechanisms by developing plastic hinges in the joints between voissoirs. This behavior is in good agreement with the one exhibited by real masonry arches. Moreover, it is possible to model in an accurate way the backfill, that can be deformable. The performed analyses regarding the role played by the infill shows a clear stabilizing effect of the backfill, a decrease of the nonlinear behavior and an increase of the collapse load multiplier, as well as a reduction of the horizontal displacements of the arch.

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