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Mortar method for 2D elastic contact problems

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Abstract

In this paper, we focus on the mortar method with a segment-to-segment approach used for connecting non-conforming and non-overlapping meshes and for the contact between two elastic bodies. We briefly review the theory, present our Matlab implementation and compare results on benchmarks. We compare our results with the analytical solution of Hertz stress and popular commercial software Ansys.

Keywords: finite element method, mortar method, non-conforming meshes, elastic contacts

1 Introduction

In civil engineering and especially in the structural analysis, the design of joints between two load-bearing members, e.g., a column and girder, is one of the most crucial ingredients of the structure design. In the case of steel structures, the most commonly used joins are welded, where the loads between two members are transferred through weld and bolt connections. Here, the shear and the thrust are carried by the bolts while the compression is transferred through the contact pressure, usualy between two steel plates. Design of such joints are well described in european standards called Eurocodes [1], which provide general design instruction and methods to calculate load bearing capacity as well as rotation stiffness of the joints. Although the methods described in Eurocodes are suitable for many variations of joins, they are also in general time consuming. With the introduction of softwares using finite element method (FEM) [3] there has been a shift from Eurocodes to more universal models using shell elements and beams elements. In many engineering applications, one of the commonly used methods for simulation of contact pressure between two bodies is the introduction of artificial rigid non-linear thrust beams with additional inequality boundary conditions to allow only compression forces. This approach yields satisfactory results in situations, where two bodies are initially in very close proximity and the influence of friction is negligible. A different approach is to use the so-called mortar method [11] which creates a way of connecting non-conforming meshes and allows contacts between two elastic bodies. In this paper, we present the basic idea of mortar methods for linear elasticity contact problems in 2D, our implementation in Matlab [12] using fully vectorized Matlab implementation of elastic problems [4] and a solution of selected benchmarks compared with widely-used popular commercial software Ansys [2].

The paper is organized as follows; we start with a short review of the mathematical background; see Section 2 for the derivation of the algebraic formulation of the corresponding optimization problem and Section 3 for the introduction of the Mortar method. Section 4 presents considered benchmarks and our numerical results. The final Section 5 concludes the paper and presents our future work.

2 Algebraic formulation of the contact problem for elastic bodies

We consider two bodies denoted as $\Omega^1,\ \Omega^2 \subset \mathbb{R}^2$ with the possible contact defined by the frictionless contact boundary condition. We assume that the bodies are fixed on the parts of the boundaries $\Gamma_U^1,\ \Gamma_U^2 \neq \emptyset$. The load is represented by surface (prescribed on the boundaries parts $\Gamma_N^1,\ \Gamma_N^2$) and volume forces.

For the numerical solution of the proposed problem, we adopt the commonly used finite element method (FEM), see, e.g., [7], [6]. The FE partition will be denoted as $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ and consists of elementary elements. In particular, displacement fields are approximated by continuous, piece-wise linear functions and strain (stress) fields are approximated by piece-wise constant functions. The final discretized problem can be classified as an optimization problem with simple equality and inequality constraints.

The algebraic formulation of the problem is related to the contact of two bodies. This means that an unknown displacement vector $\mathbf{v} \in \mathbb{R}^n$ consists of two parts, i.e., it has the following structure:

$$\mathbf{v} = \left(\mathbf{v}_1^T,\, \mathbf{v}_2^T\right)^T ,$$

where $\mathbf{v}_i$ denotes the displacement vector on $\Omega^i, i = 1, 2$. We define the space

$$\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{B}_E\mathbf{v} = \mathbf{o}\}, \quad (1)$$

and the set of feasible displacements

$$\mathcal{K} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{B}_E\mathbf{v} = \mathbf{c}_E,\ \mathbf{B}_I\mathbf{v} \le \mathbf{c}_I\}. \quad (2)$$

Here the equality constraint matrix $\mathbf{B}_E \in \mathbb{R}^{m_E \times n}$ represents the Dirichlet boundary conditions defined on $\Gamma_U^1,\ \Gamma_U^2$. The inequality constraint matrix $\mathbf{B}_I \in \mathbb{R}^{m_I \times n}$ represents the non-penetration condition on the contact zones $\Gamma_N^1,\ \Gamma_N^2$. Notice that $\mathcal{K}$ is convex and closed.

Let $\mathbf{K} \in \mathbb{R}^{n \times n}$ be a block diagonal matrix consisting of the elastic stiffness matrices $\mathbf{K}^i$ defined on each domain $\Omega^i,\ i = 1, 2$. Due to the presence of the Dirichlet boundary conditions on both domains and the Korn inequality, we can define the energy norm on $\mathcal{V}$:

$$\|\mathbf{v}\|_e := \sqrt{\mathbf{v}^T\mathbf{K}\mathbf{v}} = \sqrt{\sum_{i=1}^{2} \mathbf{v}_i^T\mathbf{K}^i\mathbf{v}_i},\ \mathbf{v} = \left(\mathbf{v}_1^T,\ \mathbf{v}_2^T\right)^T \in \mathcal{V}.$$

Notice that using this norm is suitable both from mechanical and mathematical points of view.

The algebraic formulation of the contact elastic problem can be written as the following optimization problem

$$\text{Find } \mathbf{u} \in \mathcal{K} : J(\mathbf{u}) \le J(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{K}, \quad (3)$$

where

$$J(\mathbf{v}) = \sum_{i=1}^{2} \frac{1}{2}\mathbf{v}_i^T\mathbf{K}^i\mathbf{v}_i - \mathbf{f}_i^T\mathbf{v}_i. \quad (4)$$

Here $\mathbf{v}$ and $\mathbf{f}$ denote the discretized domain displacements and the discretized domain vector of prescribed forces.

3 Mortar method

To describe contact between $\Gamma_N^1,\ \Gamma_N^2$, we divide contact zones into master-slave surfaces $\gamma_c^{(1)}$,$\gamma_c^{(2)}$ in such a way, that for every master surface $\gamma_c^{(1)}$ on Γ_N^1 there exists slave surface $\gamma_c^{(2)}$ on Γ_N^2. Between each master-slave contact surfaces, we can define discretized contact energy Π_c caused by traction forces $\mathbf{t} \in \mathbb{R}^{m_I}$ on gap $\mathbf{g} \in \mathbb{R}^{m_I}$ between those surfaces

$$\Pi_c(\mathbf{v}, \mathbf{t}) = \mathbf{t}^T\mathbf{g} \quad (5)$$

with Karush–Kuhn–Tucker (KKT) conditions for friction-less contact

$$g_j \ge 0, \quad (6)$$

$$t_j \leq 0, \tag{7}$$

$$t_j g_j = 0. \tag{8}$$

Inequalities (6) ensure non-penetrating of bodies, inequalities (7) enforce the pressure exclusively on the interface, and complementarity equations (8) secure pressure to occur if and only if two bodies are in contact with zero gaps as well as prevent contact pressure to act in the case of a non-zero gap. Rewriting gap constrain (6) to the matrix representation, we obtain mortar inequality constrain

$$\mathbf{B}_I \mathbf{v} \leq \mathbf{c}_I \tag{9}$$

with mortar constrain matrix $\mathbf{B}_I$ and mortar constrain vector $\mathbf{c}_I$ given by

$$\mathbf{B}_I = \mathbf{N}^T(\mathbf{D} - \mathbf{M}), \tag{10}$$

$$\mathbf{c}_I = -\mathbf{N}^T(\mathbf{D} - \mathbf{M})\mathbf{x}, \tag{11}$$

where $\mathbf{x} \in \mathbb{R}^n$ is node coordinate vector and $\mathbf{N} \in \mathbb{R}^{2m_I \times m_I}$ is a block-diagonal matrix with diagonal blocks of the normal vector from each slave node n_{sl} on slave surface $\gamma_c^{(1)}$,

$$\mathbf{N} = \begin{bmatrix} \mathbf{n}_1 & & \\ & \ddots & \\ & & \mathbf{n}_{sl} \end{bmatrix}.$$

Matrices $\mathbf{M}, \mathbf{D} \in \mathbb{R}^{2m_I \times n}$ are composed from rows representing dependencies of the movement between master and slave nodes with components given by

$$\mathbf{D}_{j,j} = D_{j,j}\mathbf{I}_2 = \int_{\gamma_c^{(1)}} N_j^{(1)} d\gamma \mathbf{I}_2, \tag{12}$$

$$\mathbf{M}_{j,l} = M_{j,l}\mathbf{I}_2 = \int_{\gamma_c^{(1)}} \Phi_j N_l^{(1)} d\gamma \mathbf{I}_2. \tag{13}$$

Here, N_j and N_l denote basis functions of the slave and master mortar elements, respectively, Φ_j denotes dual basis functions of slave mortar element and $\mathbf{I}_2$ a 2×2 identity matrix. Both integrals are evaluated over the slave surface $\gamma_c^{(1)}$. The dimension of mortar elements is always one lower than the dimension of the elements (in our example we have 2D elements and 1D mortar elements) with the same basis functions as the normal elements. In our benchmarks, we consider linear elements with given explicit dual basis functions

$$\Phi_1 = 1/3(1 - 3\zeta), \qquad \Phi_2 = 1/2(1 + 3\zeta).$$

For higher order basis function, please refer to [11].

Additionally, we have to mention some technical details about mortar constraints. Each integration in (12),(13) is further multiplied by identity matrix $\mathbf{I}_2$ since there are two degrees of freedom (DOF) for each node. Further, normal vectors and node positions are based on the deformation of bodies, thus the inequality (9) is non-linear in $\mathbf{v}$, see next subsection and (14). The last equation $\mathbf{D} - \mathbf{M}$ describes the mutual movement of slave-master nodes respectively, which causes the sum of each row of $\mathbf{D} - \mathbf{M}$ to be zero. In special case, when all nodes from $\gamma_c^{(1)}$ lie on $\gamma_c^{(2)}$, the equation (11) would evaluate zero vector. In such case changing inequality in (9) to equality ensures the glue boundary condition of two non-conforming meshes.

Since N_j, N_l, and Φ_j are well defined, the only missing methodology component is the evaluation of the slave surface $\gamma_c^{(1)}$. This process is described in the following subsection.

3.1 Segmentation

To calculate the slave surface area $\gamma_c^{(1)}$, we use the segmentation algorithm. For simplicity, let us introduce a case with one slave and two master segments as demonstrated in Figure 1.

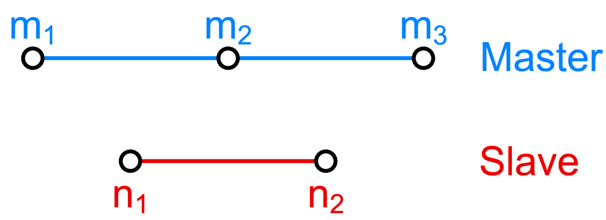

Figure 1: Master-slave elements in the mortar method.

During the preparation phase, we compose the incidence matrix [8] $\mathbf{C}$ describing connectivity between mortar elements and nodes. Afterwards, we evaluate normal vectors for each node as an average over each connecting mortar element. Using the normal vector and the elements for each slave element, we create a polygon. This structure is used to determine if the nodes from the master elements are present, see Figure 2. For each identified node, we determine its connected mortar elements using the incidence matrix and we define master-slave pairs for further evaluation.

One of the methods for the calculation of bounds of slave surface is based on the nodes projection onto the surface of mortar element. In our implementation, we calculate each node $\mathbf{p}$ using linear interpolation of element nodes coordinates $\mathbf{x}_{i,1}, \mathbf{x}_{i,2}$ in combination with linear interpolation of normal vectors from each element node $\mathbf{n}_{i,1}, \mathbf{n}_{i,2}$, i.e.,

$$\mathbf{p} = \mathbf{x}_{i,1} + \alpha(\mathbf{x}_{i,2} - \mathbf{x}_{i,1}) + \beta\left(\mathbf{n}_i + \alpha(\mathbf{n}_{i,2} - \mathbf{n}_{i,1})\right). \tag{14}$$

Since the equation (14) is not linear, we adopt Newton method [9] to calculate the coefficient $\alpha \in [0, 1]$, which is used to define boundary of slave surface $\gamma_c^{(1)}$ and

$$\mathbf{x}_{i,1} + \alpha(\mathbf{x}_{i,2} - \mathbf{x}_{i,1})$$

defines point on mortar element, see Figure 3. Notice that in the case when α is not in the feasible set defined by interval $[0, 1]$, the respective node cannot be projected onto the element. This property can be further exploited to decrease the number of Newton method iterations.

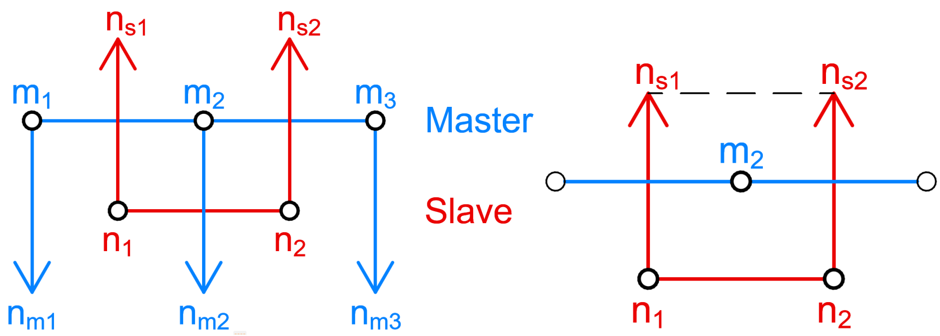

Figure 2: Segments normal vectors (left) and search for nodes in polygon (right).

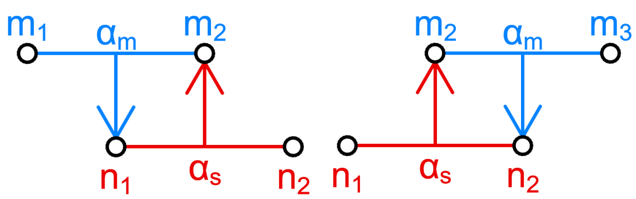

Figure 3: Master-slave projection onto two separated master-slave pairs.

Based on the convergence of the Newton method for both master and slave segments, it is possible to determine the mutual position of the two elements as well as their contact surface. Using this process for master-slave pairs gives the required integration boundary for (12),(13) defined as intervals $\alpha_s = [\alpha_{s1}, \alpha_{s2}]$ and $\alpha_m = [\alpha_{m1}, \alpha_{m2}]$ for slave and master element, respectively.

4 Numerical benchmarks

As a benchmark, we consider a contact between two bodies Ω^1, Ω^2 of homogeneous elastic material with zero displacement on boundary Γ_U^2 and imposed displacement $\mathbf{u_z}$ on Γ_U^1, see Figure 4. Material of Ω^1 is defined by Young's modulus $E = 200$ GPa and Poisson's ratio $\nu = 0.3$. For our benchmark, the material of Ω^2 was chosen to behave as rigid with Young's modulus $E = 160$ TPa and Poisson's ratio $\nu = 0.3$. Body Ω^1 is a semicircle with radius of $r = 8$ mm and body Ω^2 is a rectangular with cross-section of $30/10$ mm. These blocks have prescribed contact zones Γ^1_m, Γ^2_s which also describe master and slave side of contact region respectively. Displacement is applied on face of the block Ω^1, spread equally across whole edge. Block Ω^1 is discretized by 2621 finite element, block Ω^2 is discretized by 7500 finite element, see Figure 5.

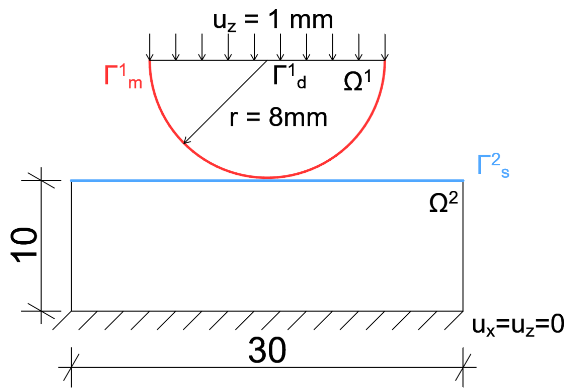

Figure 4: Geometry of benchmark.

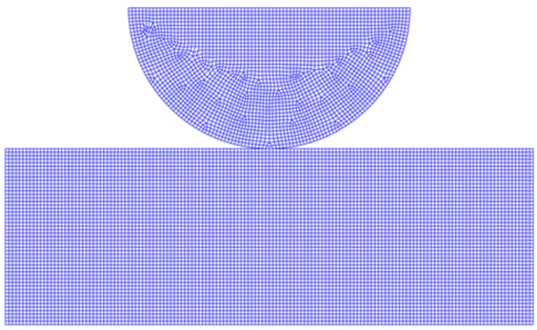

Figure 5: Mesh of benchmark with 2621 elements in semicircle and 7500 elements in rectangle.

Benchmark presented in this section has an analytical solution for contact pressure known as Hertz stress [10] with maximum normal contact pressure

$$p_c = \frac{4rp}{\pi b}, b = 2\sqrt{\frac{2r^2p(1-v^2)}{E\pi}}.$$

Here p_c is a contact pressure, r is the radius of the semicircle, p is a uniform load on the top of the semicircle and b is the width of the contact surface after deformation, see Figure 6. Comparison of contact stresses related to Hertz stress is shown in Table 1. Even though there is not an exact match between Hertz stress and results computed by our implementation, it is still considered to be a valid solution for engineering applications. Since the Ansys solution has a sufficient match with the Hertz solution, we further compare element stresses with the solution from Ansys, see Figure 7 and Table 2.

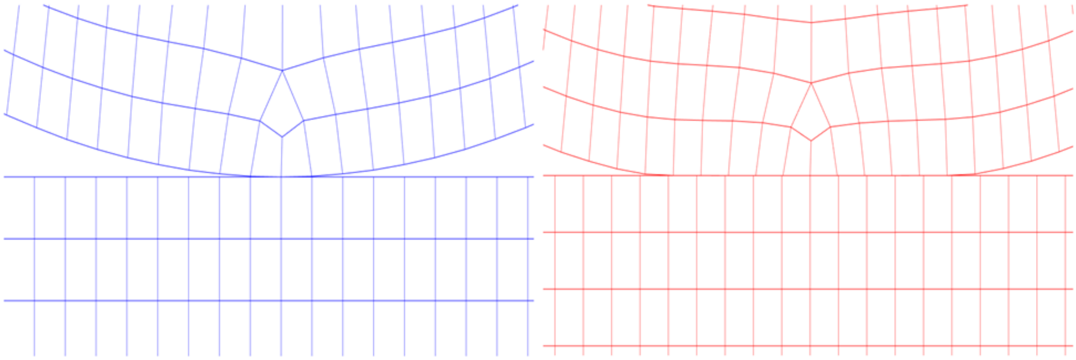

Figure 6: Undeformed (left) and deformed (right) mesh in contact region

Approach	p_c [Mpa]	Abs. Diff [Mpa]	Rel. Diff [%] [2pt]
Hertz	14003	-	**-**
Ansys	13637	367	**2.62**
Our	13159	845	**6.03**

Table 1: Comparison of maximal normal contact pressure between Hertz stress, Ansys solution and our solution.

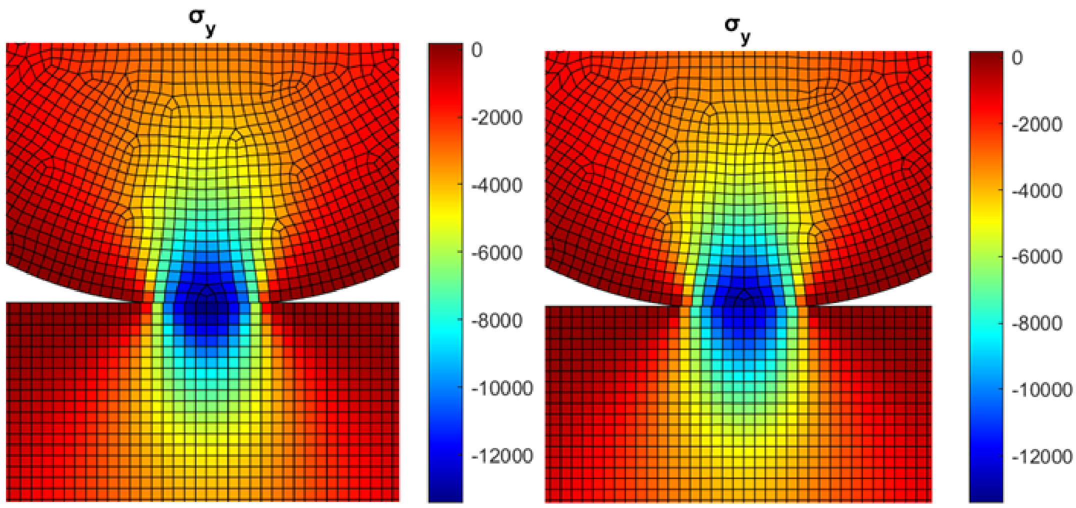

Figure 7: Comparison between our stress (left) and stress computed in Ansys (right).

Stress	Matlab [Mpa]	Ansys [Mpa]	Abs. Diff [Mpa]	Rel. Diff [%] [2pt]
$\sigma_{y,max}$	190	187	3	**1.58**
$\sigma_{y,min}$	-13159	-13420	261	**1.94**
$\sigma_{x,max}$	700	664	36	**5.42**
$\sigma_{x,min}$	-11305	-10221	1084	**10.61**
$\tau_{xy,max}$	3270	3437	167	**4.86**

Table 2: Comparison of maximal and minimal stresses between our solution and Ansys for Hertz problem.

We implement segment-to-segment mortar method for contact between two elastic bodies without friction in Matlab and compare our solution against the analytical Hertz stress as well as results from commercial software Ansys. Comparing different results for normal contact stresses in Table 1, we are able to achieve relative difference 6% from the Hertz stress. Further in Table 2, we compared element stresses with relative difference of 1.94% for element stress $\sigma_{y,min}$ at the contact area.

5 Conclusion and future work

In the paper, we present the results of our implementation of the mortar method for solving the elasticity contact problem between two bodies discretized by FEM. Based on the comparison with commercial software, the results confirm our approach to be valid.

In our future work, we expand the presented approach to contact problems with elastoplastic behaviour. This challenging task requires efficient numerical solvers, therefore we adopt optimal quadratic programming solvers [6], such as the Semi-monotonic Augmented Lagrangian method (SMALSE-M) and Modified Proportioning with Reduced Gradient projections (MPRGP). To improve the performance even further and extend the possibility of solving large-scale problems on parallel architectures, we apply total finite element tearing and interconnecting (TFETI) [5], which is a robust non-overlapping domain decomposition method. This approach grants a possibility to consider 3D contact problems where a large number of FE nodes can easily exceed the number of unknowns in corresponding optimization problems to millions. Additionally, we have to implement the extension of the mortar method to 3D problems to solve the contact problems with non-conforming meshes.

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