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Form-finding of reticulated shells for a given plan layout with geometric constraints

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Abstract

A numerical tool is implemented to address the design of reticulated shells through funicular analysis. As discussed in the literature, the force density method can be conveniently implemented to cope with the equilibrium of funicular networks, using independent sets of branches in the case of grids having fixed plan projection. In this contribution, optimal networks are sought not only in terms of an independent set of force densities, but also in the vertical coordinates of the restrained nodes. Constraints are enforced on the coordinates of the nodes, to prescribe a feasible design domain, and on the geometry of the members, to control their length and inclination with respect to a given reference direction. Due to its peculiar form, the arising multi-constrained problem can be efficiently solved through techniques of sequential convex programming that were originally conceived to handle formulations of size optimization for elastic structures. Networks that are fully feasible with respect to the enforced local constraints are retrieved in a limited number of iterations, with no need to initialize the procedure with a feasible starting guess. The same algorithm applies to general networks with any type of geometry and restraints.

Keywords: funicular analysis, form-finding, force density method, structural optimization, mathematical programming, geometric constraints

1 Introduction

Reticulated shells take their strength from their double curvature, while consisting of members that mainly undergo axial forces [1, 2]. Funicular analysis is extensively adopted to deal with the design of arcuated structures, see e.g. [3, 4]. Following this approach, spatial structures such as three-dimensional trusses and reticulated shells can be modelled as statically indeterminate networks of vertices and edges of prescribed topology. Boundary supports are given at the restrained nodes of the network, where reactions may arise; unrestrained ones are in equilibrium with the applied vertical and horizontal loads. The equations governing the equilibrium of the nodes can be conveniently written in terms of the horizontal reaction, see in particular [5], or in terms of the force densities, i.e. the ratio of force to length in each branch of the network [6]. As investigated in the literature, independent sets of branches can be detected for networks with fixed plan geometry, see in particular [7]. However, enforcing a prescribed range of heights for the nodes is not a trivial task from a numerical point of view. To this goal a numerical tool has been presented in [8], implementing a multi-constrained minimization problem. Both the independent force densities and the vertical coordinates of the restrained nodes were used as unknowns, whereas suitable norms of the horizontal thrusts, i.e. the horizontal components at the restrained nodes, were adopted as objective function. Due to its peculiar form, this problem can be efficiently solved through techniques of sequential convex programming that were originally conceived to handle large scale multi-constrained formulations of size optimization for elastic structures, see in particular [9]. In a stress-based minimum weight problem of truss design, the area of the sections is sought such that the weight is minimized, for given strength limits. In a statically determinate truss, the objective function is linear in the unknowns, whereas the stress depends on the inverse of the unknowns. Dealing with the funicular polygon of a set of vertical loads, it may be shown that: i) the thrust is linear in the only independent force density; ii) the height of the unrestrained nodes is linear in the vertical coordinate of the restrained ones, while depending on the the inverse of the independent force density. Methods of sequential convex programming are available that implement approximations of the objective functions and constraints in the direct or the reciprocal variable according to the sign of the gradient [10]. These gradient-based methods can be conveniently adopted to handle both minimization problems. Extensions of the approach in [8] have been proposed in [11], considering multi-layer networks in the funicular analysis of vaults, and in [12], addressing overhang constraints in form-finding of gridshells. In both cases, the thrust was used as objective function.

In this contribution, the research of the optimal shape of reticulated shells is made by adopting an alternative objective function, i.e. the Maxwell number, which is the sum of the force-times-length products for all the branches in the spatial network. According to Maxwell's theorem [13], this number equals the sum of the load-times-distance products for all the forces acting upon the network (the distance being measured from an arbitrary origin to the point of application of the load), see also discussions and examples in [14, 15]. Constraints are of geometric type, being related to the maximum inclination and length of the members and to the allowed range for the height of the nodes of the spatial networks. In the next sections, a short overview of the force density method is given, as for the multi-constrained formulation. A numerical examples is shown to demonstrate the method and draw preliminary conclusions.

2 Force density method

The "force density method" [6] is used to cope with the equilibrium of spatial networks. A funicular network consists of $n_s = n + n_f$ nodes and m branches, which undergo axial forces only. The axes of the Cartesian reference system with origin O are denoted by x, y, and z. Hence, $\mathbf{x}_s$, $\mathbf{y}_s$, $\mathbf{z}_s$ are vectors gathering the coordinates of the n_s nodes: $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ refer to the n unrestrained nodes, i.e. the nodes subject to external forces; $\mathbf{x}_f$, $\mathbf{y}_f$, $\mathbf{z}_f$ collect the n_f restrained nodes, i.e. those where reactions arise. The connectivity matrix that fully describes the shape of the grid is $\mathbf{C}_s$, having subset $\mathbf{C}$ for the unrestrained nodes and $\mathbf{C}_f$ for the restrained ones. The vectors that collect the coordinate difference of the nodes along the axis x, y, z are denoted by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, respectively:

$$\mathbf{u} = \mathbf{C}_s\mathbf{x}_s, \quad \mathbf{v} = \mathbf{C}_s\mathbf{y}_s, \quad \mathbf{w} = \mathbf{C}_s\mathbf{z}_s. \tag{1}$$

The force densities, i.e. the ratios force to length for each branch of the network, are stored in $\mathbf{q} = \mathbf{L}^{-1}\mathbf{s}$, being $\mathbf{s}$ the vector that collects the forces in the m branches. The length of the branches $l_i = \sqrt{u_i^2 + v_i^2 + w_i^2}$ is gathered in the square matrix $\mathbf{L} = \text{diag}(\mathbf{l})$. Only gravity loads are considered in this study. Vertical point forces are prescribed at the unrestrained nodes through vector $\mathbf{p}_z$. Due to the introduction of the vector $\mathbf{q}$, the equilibrium of the unrestrained nodes is given by a set of linear equations that are uncoupled in the three axes. As discussed in [7, 11, 14], the horizontal equilibrium of the loaded nodes in networks with fixed plan projection reads:

$$\begin{bmatrix} \mathbf{C}^T\text{diag}(\mathbf{C}_s\mathbf{x}_{s0}) \\ \mathbf{C}^T\text{diag}(\mathbf{C}_s\mathbf{y}_{s0}) \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{2}$$

where the vector $\mathbf{x}_{s0}$ and $\mathbf{y}_{s0}$ gather the fixed x and y coordinate of the nodes, respectively. Eqn. (2) implies that $m - r$ independent force densities $\overline{\mathbf{q}}$ can be retrieved, being r the rank of the coefficient matrix. The r dependent force densities $\widetilde{\mathbf{q}}$ may be written as:

$$\widetilde{\mathbf{q}} = \mathbf{B}\overline{\mathbf{q}} + \mathbf{d}, \tag{3}$$

where $\mathbf{B}$ and $\mathbf{d}$ are found by applying Gauss-Jordan elimination to Eqn. (2), see [14]. The equilibrium along the z axis reads:

$$\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{z} + \mathbf{C}^T\mathbf{Q}\mathbf{C}_f\mathbf{z}_f = \mathbf{p}_z, \tag{4}$$

being $\mathbf{Q} = \text{diag}(\mathbf{q})$.

3 Optimization problem

A multi-constrained minimization is stated in terms of any reduced set of independent force densities $\overline{\mathbf{q}}$ and of the vertical coordinates of the restrained nodes $\mathbf{z}_f$. It reads:

$$\begin{cases} \min\limits_{\overline{\mathbf{q}},\ \mathbf{z}_f} f & \text{(5a)} \\ \text{s.t. } \widetilde{\mathbf{q}} = \mathbf{B}\overline{\mathbf{q}} + \mathbf{d}, & \text{(5b)} \\ \quad \mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{z} + \mathbf{C}^T\mathbf{Q}\mathbf{C}_f\mathbf{z}_f = \mathbf{p}_z, & \text{(5c)} \\ \quad \left(\dfrac{\tan\alpha_i}{\tan\alpha_{max}}\right)^2 \leq 1 \quad \text{for } i = 1...m, & \text{(5d)} \\ \quad \left(\dfrac{l_i}{l_{max}}\right)^2 \leq 1 \quad \text{for } i = 1...m, & \text{(5e)} \\ \quad z_j(\overline{\mathbf{q}}, \mathbf{z}_f) \geq z_j^{min} \quad \text{for } j = 1...n, & \text{(5f)} \\ \quad z_j(\overline{\mathbf{q}}, \mathbf{z}_f) \leq z_j^{max} \quad \text{for } j = 1...n, & \text{(5g)} \\ \quad \widetilde{q}_k \leq 0 \quad \text{for } k = 1...r, & \text{(5h)} \\ \quad \overline{q}_i \leq 0 \quad \text{for } i = 1...m-r, & \text{(5i)} \\ \quad z_{f\,h}^{min} \leq z_{f\,h} \leq z_{f\,h}^{max} \quad \text{for } h = 1...n_f. & \text{(5j)} \end{cases}$$

In the above statement, the objective function accounts for the sum of the force-times-length products computed in each branch of the network [13]. Since anti-funicular networks are dealt with, see Eqns. (5h–5i), one has:

$$f = -\mathbf{s}^T\mathbf{l} = -\mathbf{q}^T\mathbf{L}^2. \tag{6}$$

Eqn. (5b) allows recovering the set of dependent force densities $\widetilde{\mathbf{q}}$ from the vector of the independent ones $\overline{\mathbf{q}}$. Eqn. (5c) states the equilibrium of the unrestrained nodes in the vertical direction, to compute $\mathbf{z}$ from $\overline{\mathbf{q}}$ and $\mathbf{z}_f$.

Denoting by $\tan\alpha_i$ the tangent of the angle between the direction of the i-th bar and a reference direction, one has:

$$\tan\alpha_{i\,x} = \sqrt{\frac{v_i^2 + w_i^2}{u_i^2}}, \text{ or } \tan\alpha_{i\,y} = \sqrt{\frac{u_i^2 + w_i^2}{v_i^2}}, \text{ or } \tan\alpha_{i\,z} = \sqrt{\frac{u_i^2 + v_i^2}{w_i^2}}, \tag{7}$$

in case the reference direction to compute the inclination is aligned with the x, y, or z axis, respectively. Hence, Eqn. (5d) is used to prescribe the maximum value of the deviation of the inclination of each branch of the network from the reference one (α_{max}). Eqn. (5e) enforces the maximum length of the branches of the network (l_{max}). The coordinate difference of the connected points given in Eqn. (1) are used to enforce these geometric constraints in a straightforward way.

Eqns. (5f–5g) are two sets of inequalities that prescribe lower and upper limits for $\mathbf{z}$. The design domain is such that each one of the n coordinates z_j must be bounded from below by z_j^{min} and from above by z_j^{max}. Similarly, Eqn. (5j) deals with side constraints for the minimization unknowns $z_{f\,h}$, enforcing lower and upper limits for the restrained nodes.

The whole statement is solved by means of the Methods of Moving Asymptotes [13], see the discussion in Section 1. Being MMA a first order method, the sensitivity of the objective functions and constraints are needed, see e.g. in [8].

4 Numerical example

A rectangular bay with overall size $d_x = 5.50\,\text{m} \times d_y = 3.62\,\text{m}$ is addressed. In the (given) projection onto the horizontal plane, branches have length $l_{xy} = 0.15\,\text{m}$, with a reciprocal angle of $60°$ or $120°$. Fully restrained nodes are assumed along the perimeter, as well as reference nodal forces equal to $1\,\text{N}$ acting along the z axis, i.e. the vertical one, all over the network.

Exploiting symmetry, only one forth of the bay is considered in this preliminary investigation. By applying Gauss-Jordan elimination to the system governing the horizontal equilibrium, see Eqn. (2), it is found that $r = 370$ dependent force densities exist, out of $m = 391$ branches, meaning that only $m - r = 21$ independent force densities exist. Indeed, the optimization problem of Eqn. (5) is set up in terms of 43 minimization unknowns, i.e. 21 force densities and 22 vertical coordinates of the nodes that are restrained along the z axis. The lower bound and the upper bound of the height of the nodes is set to $2.5\,\text{m}$ and $3.5\,\text{m}$, respectively.

At first, a problem that disregards the geometric constraints of Eqns. (5d) and (5e) is investigated. The optimal reticulated shell, along with a map of the forces acting in the branches, is represented in Figure 1(a). In all the pictures that follow, the symbols $+$ and $\circ$ stand for points where the nodes of the network touch the upper bound and the lower bound of the prescribed design region, respectively.

In some approaches of additive manufacturing, see in particular the discussion in [12], it is quite frequent to fabricate a few parts separately and, then, assembling the components into a final complex structure. A possible way to produce the designed quarter of a reticulated grid consists in orienting the part, during the fabrication process, such that the y axis is aligned to the vertical direction. In Figure 1(b), the optimal design found accounting for overhang constraints enforcing $\alpha_{max} = 45°$, see Eqn. (5d), is given. Finally, in Figure 1(c), the optimal design retrieved enforcing both $\alpha_{max} = 45°$ and a maximum length for the branches equal to $l_{max} = 0.165\,\text{m}$, see Eqn. (5e), is provided.

All the simulations are characterized by full feasibility of the enforced sets of constraints. Convergence has been found in less than a few tens of iterations, independently on the adopted starting guess.

Comparing the achieved optimal shapes and the stress regime in the branches, it may be pointed out that geometric constraints are responsible for major modification with respect to solutions found when dropping such kind of manufacturing limitations.

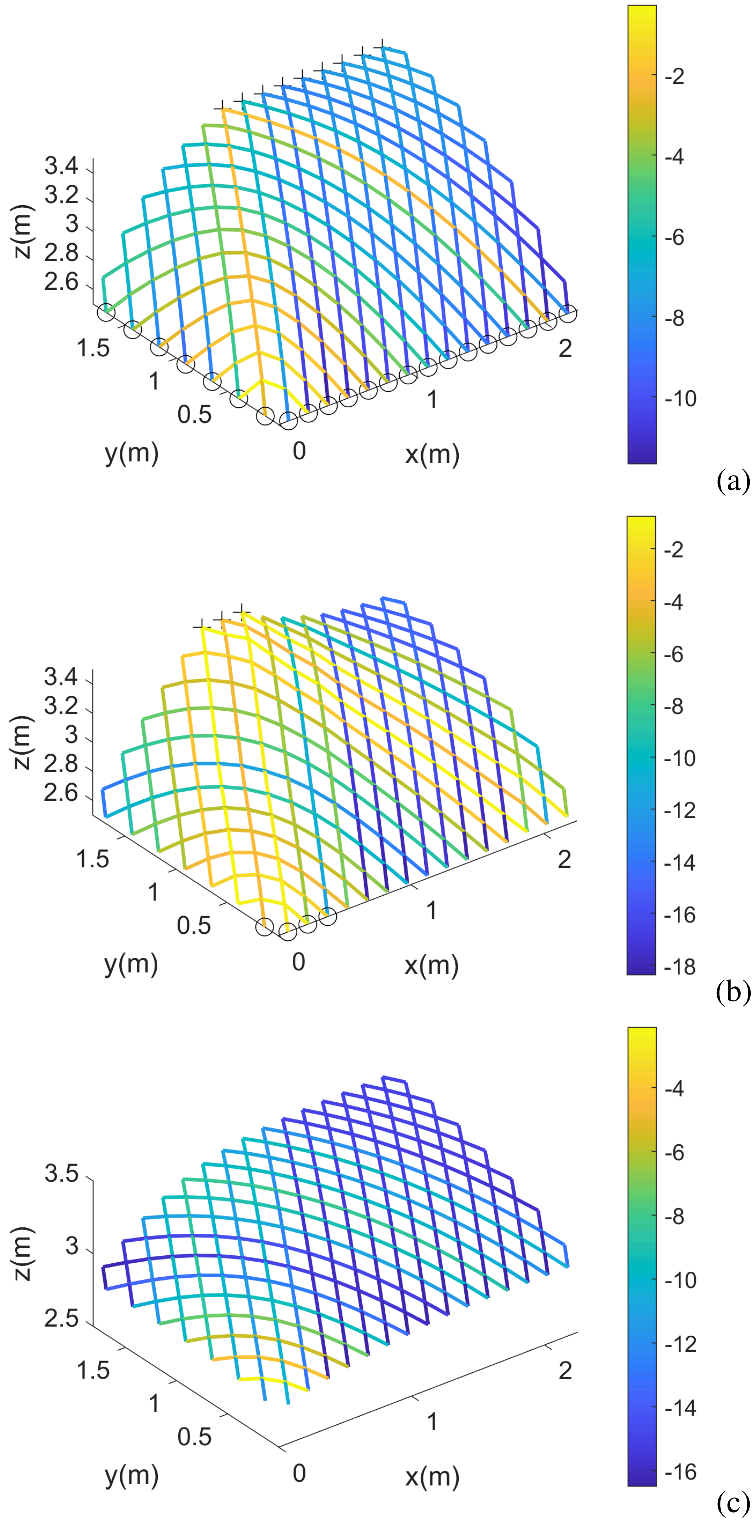

Figure 1: A rectangular bay. Optimal network and element forces, in N, for minimum magnitude of the Maxwell number: without geometric constraints (a), including overhang constraints (b), with both overhang and length constraints.

5 Concluding remarks

In this contribution, a numerical tool has been implemented to address the design of reticulated shells through funicular analysis. As investigated in the recent literature, the force density method can be conveniently implemented to cope with the equilibrium of spatial networks of trusses, using independent sets of branches in the case of grids having fixed plan projection. In this contribution, optimal networks are sought formulating a problem of structural optimization both in terms of any independent set of force densities, and in the vertical coordinates of the restrained nodes. The Maxwell number, which is the sum of the force-times-length products for all the branches in the spatial network, is used as objective function. Constraints are enforced on the coordinates of the nodes, to prescribe a feasible design domain, and on the geometry of the members, to control their length and inclination with respect to a prescribed direction. The arising multi-constrained problem is efficiently handled by techniques of sequential convex programming.

A numerical example has been shown retrieving networks that are fully feasible with respect to the enforced local constraints. It is found that the shape of the retrieved reticulated shell is remarkably sensitive to the considered constraints. Further research includes extended tests of the procedure, investigating grids with different plan layout and accounting for probabilist loading, see [16].

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