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A stepwise Bayesian updating approach by enhancing an active learning Gaussian process regression model

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Abstract

Bayesian updating framework is regarded as a promising approach for probabilistic calibration and uncertainty quantification, and the main obstacle for practical engineering problems is the high computational cost, especially for time-consuming models. The attractive point of traditional Bayesian updating with structural reliability methods (BUS) is to reformulate Bayesian updating into a structural reliability problem by constructing the limit state function with the likelihood and an auxiliary random variable. This paper proposes a step-wise Bayesian updating approach, by developing a varying observation domain-based strategy to reduce the dimensionality and nonlinearity of the constructed limit state function. In our work, the reliability problem is decomposed into a series of sub-problems by attributing random samples of the auxiliary random variable to each sub-problem. Thereafter, the exponential relation in each limit state function degenerates into squared linear additive type, so as to comprehensively reduce the nonlinearity. To overcome the inefficiency caused by rare event, this paper further develops an active learning procedure based on Gaussian process regression (GPR) to approximate a series of induced limit states as well as the acceptance rates. The main advantage of this procedure is of sharing the common performance function evaluations which can largely reduce the computational cost.

Keywords: Bayesian updating, active learning, Gaussian process regression, metamodeling

1 Introduction

With the development of sensing technologies, various sources of data in modern mechanical systems become available. Model updating is an essential step to assimilate the experimental data into computational models so as to provide updated and more accurate model parameters for structure and infrastructure health monitoring. Bayesian updating framework is regarded as a promising approach for probabilistic calibration and uncertainty quantification, and the main obstacle for practical engineering problems is the high computational cost, especially for time-consuming models. To tackle this problem, many approaches are proposed, such as Markov Chain Monte Carlo-based method, and Bayesian updating with structural reliability methods (BUS), many of which are restricted to small-scale numerical examples. This paper aims at extending the application scope of Bayesian updating by modifying the BUS method.

Many updating approaches are proposed, such as Markov Chain Monte Carlo-based method, and Bayesian updating with structural reliability methods (BUS) [1], many of which are restricted to small-scale numerical examples [2]. This paper aims at extending the application scope of Bayesian updating by modifying the BUS method. The attractive point of traditional BUS methods is to reformulate Bayesian updating into a structural reliability problem by constructing the limit state function with the likelihood and an auxiliary random variable. This reformulation makes it possible to insert all the most advanced strategies in structural reliability analysis into Bayesian updating framework to achieve higher accuracy and efficiency. However, the constructed reliability problem also introduces new problems. One significant problem is involving a highly nonlinear limit state with rare event especially when the number of observations increases [3].

The attractive point of traditional BUS methods is to reformulate Bayesian updating into a structural reliability problem by constructing the limit state function with the likelihood and an auxiliary random variable. This reformulation makes it possible to insert all the most advanced strategies in structural reliability analysis into Bayesian updating framework to achieve higher accuracy and efficiency. However, the constructed reliability problem also introduces new problems. One significant problems is involving a highly nonlinear limit state with rare event especially when the number of observations increases. This paper firstly proposes the varying observation domain-based strategy to reduce the dimensionality and nonlinearity of the limit state. In our work, the reliability problem is decomposed into a series of sub-problems by attributing random samples of the auxiliary random variable to each sub-problem. Thereafter, the exponential relation in each limit state function degenerates into squared linear additive type, so as to comprehensively reduce the nonlinearity.

Motivated by the group of most advanced structural reliability methods, this paper develops active learning procedure based on Gaussian process regression (GPR) to approximate a series of varying observation boundaries. Thereafter, the exponential relation in each limit state function degenerates into squared linear additive type, so as to comprehensively reduce the nonlinearity.

2 Problem statement of Bayesian updating

Let $y = g(\boldsymbol{x})$ denote the computer model corresponding to the engineering problem, the vector $x = (x_1, \cdots, x_n)^{\mathrm{T}}$ represents the n uncertain parameters to be updated, y is model response. Assuming that the n_{obs} number of experimental observations are denoted by a set $y_{obs} = (y^{(1)}, \cdots, y^{(k)}, \cdots, y^{(n_{obs})})$, each observation is a realization of model response. According to Bayes's rule, the prior probability distribution $f(\boldsymbol{x})$ can be updated by the observation set, then the posterior probability distribution can be estimated with the following expression

$$f(\boldsymbol{x} \mid \boldsymbol{y}_{obs}) = \frac{f(\boldsymbol{x}) f(\boldsymbol{y}_{obs} \mid \boldsymbol{x})}{\int f(\boldsymbol{x}) f(\boldsymbol{y}_{obs} \mid \boldsymbol{x}) \mathrm{d}\boldsymbol{x}}, \tag{1}$$

where $\int f(\boldsymbol{x}) f(\boldsymbol{y}_{obs} \mid \boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ is called the evidence, which is a normalizing constant ensuring that the posterior probability density integrates to one. $f(\boldsymbol{y}_{obs} \mid \boldsymbol{x})$ is the likelihood function representing the probability of obtaining the data set $\boldsymbol{y}_{obs}$ when the values of x is fixed at a given value. It also measures the agreement or discrepancy between the available experimental data and the corresponding numerical output predicted by the computer model [4]. In many literatures, the Approximate Bayesian Computation (ABC) method [5] is utilized to represent the full likelihood with an approximate function. A typical formulation of likelihood function is defined by introducing the deviation ε_y between computer model and observations, and the deviation is modelled with probability density function (PDF) $f_\varepsilon(\bullet)$. In practical applications, the observations are regarded to be mutually independent, so the general formulation of likelihood function is

$$f(\boldsymbol{y}_{obs} \mid \boldsymbol{x}) = \prod_{k=1}^{N_{obs}} f_k(y_{obs} \mid \boldsymbol{x}). \tag{2}$$

In this paper, the deviation is assumed to follow Gaussian distribution with zero mean and covariance matrix $\boldsymbol{R}$. Then the likelihood function can be expressed as

$$f(\boldsymbol{y}_{obs} \mid \boldsymbol{x}) = \frac{1}{(2\pi)^{n_{obs}/2} |\boldsymbol{R}|^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{y}_{obs} - g(\boldsymbol{x}))^T \boldsymbol{R}^{-1} (\boldsymbol{y}_{obs} - g(\boldsymbol{x}))\}, \tag{3}$$

where $g(\boldsymbol{x})$ is regarded as the model predictions at given value of $\boldsymbol{x}$. When the observations are mutually independent, the matrix $\boldsymbol{R}$ is in diagonal type.

3 BUS combined with active learning GPR model

For nonlinear computer models and non-Gaussian prior distribution, an analytical formula of the posterior PDF function in Equation (1) is usually not possible. From this perspective, performing the Bayesian updating through a sampling algorithm is a common and effective approach being widely used in model calibration. In this work, a competitive sampling algorithm named BUS is investigated and modified. The core idea of BUS is to transform the Bayesian updating into a structural reliability problem by constructing an auxiliary limit state function. In the following, a simple rejection sampling method (BUS-RS) is briefly reviewed to represent how to use BUS method to generate posterior samples.

First, the observation domain is defined as

$$\Omega = \left[p \le cf\left(\boldsymbol{y}_{obs} \mid \boldsymbol{x} \right) \right] = \left[p - cf\left(\boldsymbol{y}_{obs} \mid \boldsymbol{x} \right) \le 0 \right] = \left[h\left(\boldsymbol{x}, p \right) \le 0 \right] , \tag{4}$$

where p is an auxiliary standard uniform random variable in [0,1], c is a positive constant ensuring that $cf\left(\boldsymbol{y}_{obs} \mid \boldsymbol{x} \right) \le 1$. The determination of the c value is discussed in detail in Ref.[2]. The auxiliary limit state function becomes $h(x, p) = p - cf\left(\boldsymbol{y}_{obs} \mid \boldsymbol{x} \right)$, and the observation domain can be regarded as the failure domain in reliability analysis problem. Straub et al. has proved that the samples in the observation (failure) domain Ω actually follow the posterior distribution $f\left(\boldsymbol{x} \mid \boldsymbol{y}_{obs} \right)$.

Among the prosperous methods of reliability analysis, stochastic simulation methods that combine the advantage of GPR model has received considerable attention [6]. This paper integrates the advantage of active learning GPR approach into Bayesian updating sampling algorithm, providing a promising updating procedure with significantly reduced computational cost.

Consider the limit state function $h(\boldsymbol{x}, p)$ in Equation (4), let $\boldsymbol{z} = (\boldsymbol{x}, p)$ denote the $n+1$-dimensional random variables. The GPR model as an approximation of the true limit state function is denoted by $\hat{h}(\boldsymbol{z})$. Assuming it is a Gaussian process (denoted by $\boldsymbol{M}$) before training, i.e.,

$$\hat{h}(\boldsymbol{z}) \sim \boldsymbol{M}(m(\boldsymbol{z}), k(\boldsymbol{z}, \boldsymbol{z}')) , \tag{5}$$

where $m(\boldsymbol{z})$ is the mean function, and $k(\boldsymbol{z}, \boldsymbol{z}')$ is the kernel function representing the covariance between two realizations $\boldsymbol{z}$ and $\boldsymbol{z}'$. Then N_{tr} training data set $(\boldsymbol{Z}_{tr}, \boldsymbol{h}_{tr})$ is used to train this GPR model, the maximum likelihood method is commonly used for estimating the values of hyper-parameters in the mean and kernel function. Once the hyper-parameters are determined, $\hat{h}(\boldsymbol{z})$ can provide posterior prediction at a new realization $\boldsymbol{z}^*$ which is a Gaussian variable with mean $\mu_{\hat{h}}(\boldsymbol{z})$ and variance given by

$$\begin{aligned}\mu_{\hat{h}}(z) &= m(z) + k(z, \boldsymbol{Z}_{tr})^T \boldsymbol{K}^{-1}(\boldsymbol{h}_{tr} - m(z)) \\ \sigma_{\hat{h}}^2(z) &= k(z,z) - k(z, \boldsymbol{Z}_{tr})^T \boldsymbol{K}^{-1} k(z, \boldsymbol{Z}_{tr})\end{aligned}, \tag{6}$$

where $\boldsymbol{K}$ is a $(N_{tr} \times N_{tr})$ -dimensional matrix. Note that the variance $\sigma_{\hat{h}}^2(z)$ quantifies the prediction accuracy of the GPR model.

Once a problem for Bayesian updating is specified, the prior distribution, observation data as well as the likelihood function are determined, then the auxiliary limit state function in BUS method can be formulated accordingly. For this induced reliability analysis problem, active learning GPR based on crude Monte Carlo simulation is applied. After the adaptive training stops, the well-trained GPR model can be used to predict whether a new sample is located in the observation domain or not. The residual procedure for generating posterior samples is the same with rejection sampling mentioned in section 2.

4 Stepwise Bayesian updating approach: Enhanced BUS-GPR

BUS-AGPR proposed in Section 3 is a direct combination of BUS method with GPR model-based active learning approach. In this section, an innovative approach is proposed as an extension of BUS-AGPR, named varying observation domain approach (BUS-AGPR-VOD). The special design is motivated by the specific feature of the induced limit state function; it provides an effective way to further draw posterior samples with minimum number of evaluations of computer model.

By using the likelihood function expression in Equation (3), the original formula of limit state function is further derived as

$$\Omega = \left[p \le c' \exp\{-\frac{1}{2}(\boldsymbol{y}_{obs} - g(\boldsymbol{x}))^T \boldsymbol{R}^{-1}(\boldsymbol{y}_{obs} - g(\boldsymbol{x}))\} \right] \tag{7}$$

where $c' = c / ((2\pi)^{n_{obs}/2} |\boldsymbol{R}|^{1/2})$. It is obvious that p is a critical variable affecting the limit state, a certain value of p determines a certain observation domain for $\boldsymbol{x}$. From qualitative perspective, the smaller the value of p, the larger the observation domain will be, thus leading to a relatively higher acceptance rate. Since p is an auxiliary standard uniform random variable, here we propose to use the samples of p to establish a series of varying observation domains for active learning BUS algorithm.

Consider there are N_p samples of p, and the logarithmic formula is introduced on both sides in Equation (7) to reduce the nonlinearity of limit state function. Take the r-th sample $p^{(r)}$ for instance, the r-th varying observation domain is analytically derived as below

$$\begin{aligned}\Omega^{(r)} &= \left[\log(p^{(r)}) - \log(c') + \frac{1}{2}(\boldsymbol{y}_{obs} - g(\boldsymbol{x}))^T \boldsymbol{R}^{-1}(\boldsymbol{y}_{obs} - g(\boldsymbol{x})) \le 0 \right], \\ &= [t^{(r)}(\boldsymbol{x}) \le 0]\end{aligned} \tag{8}$$

where $t^{(r)}(\boldsymbol{x})$ is the r-th limit state function. Then a GPR model is constructed to approximate the n-dimensional limit state function instead of $n+1$-dimension one in Section 3. The flowchart of varying observation domain approach is shown in Figure 1.

Note that the training set T actually contains the model response samples $g(\boldsymbol{x}^{(s)})$ predicted by computer simulator. The training set can be repeatedly used for evaluating each limit state function $t^{(r)}(\boldsymbol{x})$. That means, most of the GPR models may share the same training samples with satisfied accuracy.

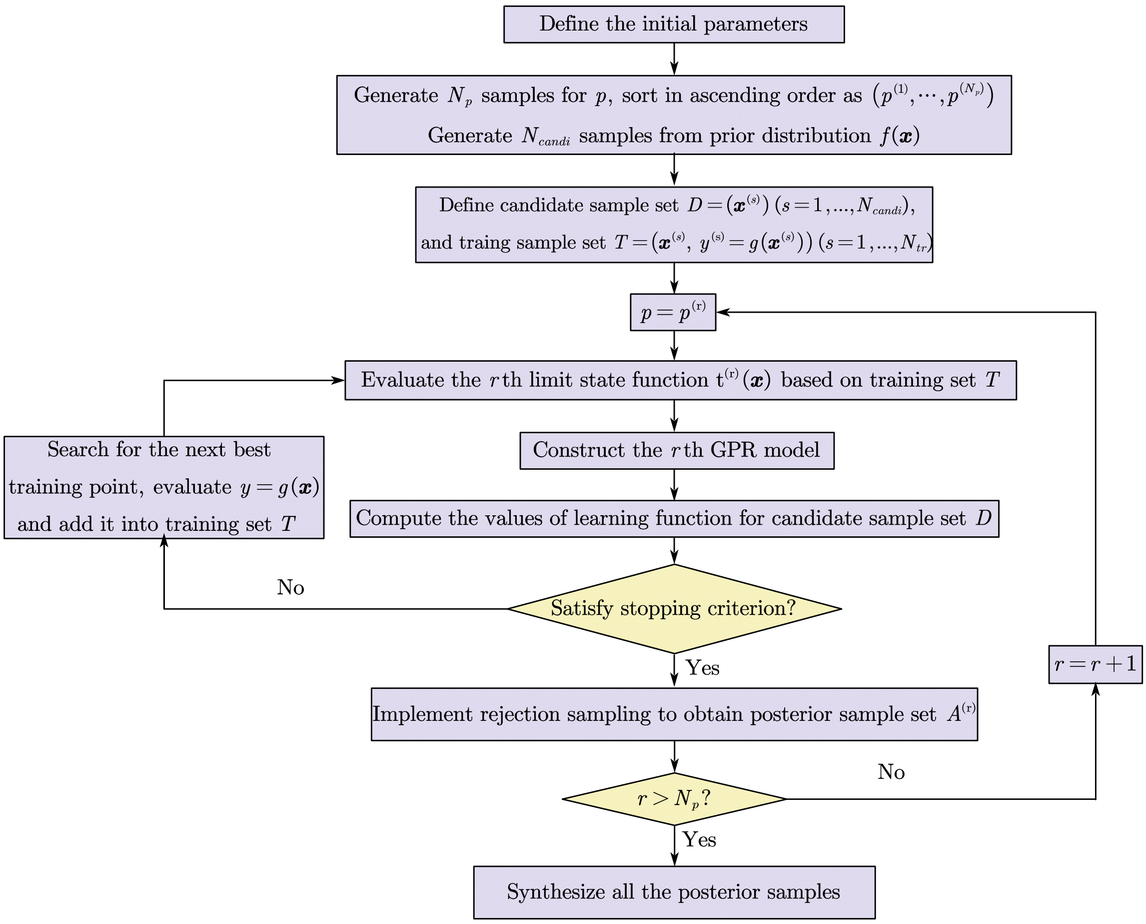


Figure 1: Flowchart of stepwise Bayesian updating approach.

When the number of observations is large, the rare acceptance rate may still make it difficult to approximate the boundary. Then other advance active learning reliability analysis approaches, such as line sampling [7] and subset simulation [8] can be combined in above flowchart. Additionally, iterative updating can be considered. That means to gradually include the observations into the auxiliary limit state function. It can partly relieve the computational burden caused by the significant decrease of acceptance rate. Furthermore, when the observations are independent, an optional modification is to utilize the strategy proposed in Ref.[4] of transforming the problem into a parallel-system reliability analysis.

5 Case studies: A cantilever beam

Recalling the typical example of cantilever beam in Ref.[3], which has an analytical solution allowing a validation of the method. The length of the beam is 5 meters under a deterministic point load $V = 20\text{kN}$ at the free end. The aim is to update the spatially flexibility function $F(x)$ based on the measured beam vertical deflections $W(x)$. The analytical formulations between the flexibility and vertical deflection are listed in detail in Ref.[3]. The prior distribution of $F(x)$ is described by a homogeneous Gaussian random field. Measurements $\boldsymbol{w}$ of the vertical deflection are made at ten points from (0.5m,1m,…,5m) along the beam with optic measurements, as shown in Figure 2.

Bayesian updating with the proposed stepwise approach as well as other comparative methods are performed. The constant c is selected as the inverse of the likelihood function at the maximum likelihood estimate. The updated flexibility functions by three methods are plotted in Figure 3(a), while the flexibilities before and after updating are drown in Figure 3(b) at two representative points $x = 1\text{m}$ and $x = 5\text{m}$, respectively. The corresponding posterior statistics of flexibility are listed in Table 1, the proposed method only consumes 67 function evaluations, showing the effectiveness and efficiency of stepwise Bayesian updating in active learning manner.

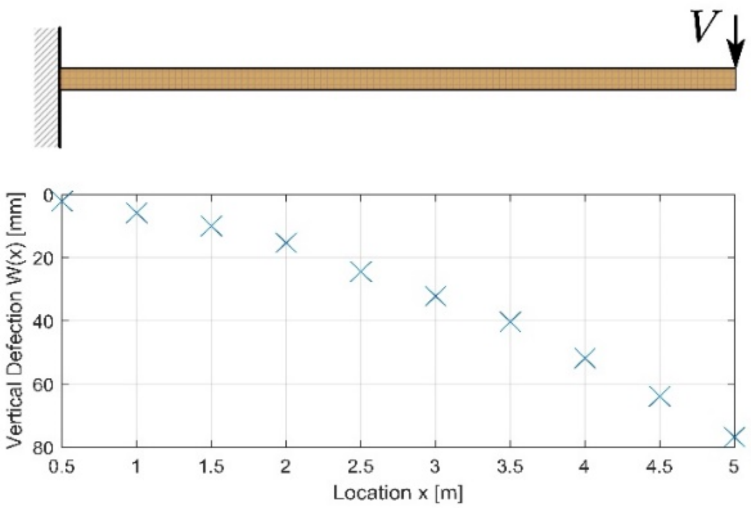
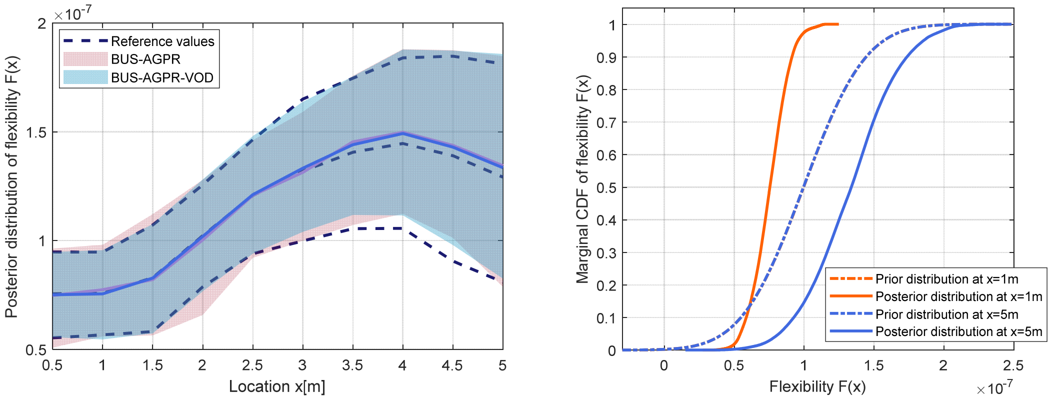


Figure 2: The vertical displacements at ten points with optic measurements.



(a) Updated flexibility function (b) Flexibility before and after updating

Figure 3: 95% CIs for BUS-AGPR and BUS-AGPR-VOD method.

Methodology	N_{call}	Posterior mean μ at X=1m (1e-7)	Posterior varianc v at X=1m (1e-16)	Posterior mean μ at X=5m (1e-7)	Posterior variance v at X=5m (1e-16)	Acceptance rate p_{acc}
BUS-AGPR	316	$0.7740^{(0.0108)}$	$1.5826^{(0.1949)}$	$1.3441^{(0.0027)}$	$9.7778^{(1.2292)}$	6.75e-4
BUS-AGPR-VOD	67	$0.7547^{(0.0002)}$	$1.5899^{(0.0275)}$	$1.3345^{(0.0003)}$	$9.6073^{(0.1608)}$	6.48e-4
True values	\	0.7606	1.7002	1.3092	9.6856	\

Table 1: The estimated posterior statistics for BUS-AGPR and BUS-AGPR-VOD method as well as the analytical results.

6 Conclusions and Contributions

Model updating is an essential step to assimilate the experimental data into computational models. In this paper, we develop a step-wise active learning procedure based on Gaussian Process Regression to approximate a series of limit state as well as the acceptance rates, in this way to largely improve the efficiency in the exploration of rare event in the induced reliability problem.

In the case study, for the multi-location type of observations, single-output metamodels such as kriging, GPR models, cannot be directly used for representing the ten-to-ten models. In this cantilever beam example, the single-output metamodels are utilized to represent the likelihood function based limit state function instead of the original computational models, hence, the single-output metamodels can still be effectively used to largely reduce the computational cost.

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References

[1] W. Betz, I. Papaioannou, D. Straub, “Adaptive variant of the BUS approach to Bayesian updating”, in: Proceedings of 9th European Conference on Structural Dynamics, 2014.

[2] W. Betz, I. Papaioannou, J.L. Beck, D. Straub, “Bayesian inference with subset simulation: strategies and improvements”, Computer methods in applied mechanics and engineering, 331: 72-93, 2018.

[3] D. Straub, I. Papaioannou, “Bayesian updating with structural reliability methods”, Journal of Engineering Mechanics, 141(3): 04014134, 2015.

[4] E. Patelli, M. Broggi, Y. Govers, J. E. Mottershead, “Model updating strategy of the DLR-AIRMOD test structure”, International Conference on Structural Dynamics, EURODYN, 2017.

[5] Z. Wang, A. Shafieezadeh, “Highly efficient Bayesian updating using metamodels: An adaptive Kriging-based approach”, Structural Safety, 84: 101915, 2020.

[6] E.C. DeCarlo, B. P. Smarslok, S. Mahadevan, “Segmented Bayesian calibration of multidisciplinary models”, AIAA Journal, 54: 1-15, 2016.

[7] J. Song, P. Wei, M. Valdebenito, M. Beer, “Active learning line sampling for rare event analysis”, Mechanical System and Signal Processing, 147: 107113, 2021.

[8] P. Wei, C. Tang, Y. Yang, “Structural reliability and reliability sensitivity analysis of extremely rare failure events by combining sampling and surrogate model methods”, Proceedings of the Institution of Mechanical Engineers, Part O: Journal of risk and reliability, 233(6), 943-957, 2019.