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Application of the Wave Finite Element method to the computation of the response of a ballastless railway track, comparison with on-site measurements

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Abstract

Dynamics of railway tracks have been studied for a long time. Many authors proposed analytical or numerical models to compute the dynamic response of the tracks. Numerical models often use the track periodicity to reduce the size of the problem. Among these numerical methods, the Wave Finite Element (WFE) method was designed to compute the dynamics of periodic structures composed of identical patterns. It was successfully applied t simplified models of railway tracks subjected to different types of loads. In these studies, tracks are modelled by periodically supported beams. In order to give access to stresses and strains at a fine scale, a much finer representation is needed. This article presents a WFE computation of the dynamics of a ballastless railway track subjected to constant moving loads. In the presented computation, the rail, the underlying slab and the support system are all represented in three dimensions. In order to validate this method, the obtained results are compared to experimental strain measurements.

Keywords: Dynamics, Vibration, Railway tracks, Moving loads, Wave Finite Element, Experimental validation.

1 Introduction

Dynamics of railway tracks have been studied for a long time. Several authors proposed analytical [1] models for this purpose. In these studies, tracks are modelled as periodically supported beams. Other authors developed numerical methods which take advantage of the track periodicity [2, 3].

Among these methods, the Wave Finite Element (WFE) method has been widely used to compute the dynamics of periodic structures and waveguides. This numerical method consists in reducing the dynamics of each spatial period (called pattern) to wave equations at its boundaries. Then, the kinematic and mechanical fields of the whole structure can be found by computing the amplitude of the waves travelling in the structure main direction.

Hoang et al. [4] successfully used the Wave Finite Element (WFE) method to compute the response of a homogeneous railway track subjected to constant moving loads. Claudet et al. [5] developed a method based on the WFE method to compute the response of railway tracks transition zones. Both Hoang et al. and Claudet et al. modelled a railway track by a periodically supported beam and the supports with mass-springs-dampers systems connected to fixed points. These simplified models can give global

results with very low computational time and strong agreements with analytical models. However, because of their simplicity, they can't give access to some values, such as stresses or strains, at a fine scale.

In this article, the WFE method is applied to the computation of the response of a ballastless railway track subjected to a constant moving load. In the computation performed, a fine three-dimensional model is used to represent the track. This model includes the rail, the support system and the underlying slab. A comparison with experimental strain measurements validates the proposed method.

After this introduction, the numerical WFE method will be presented. Then, the numerical results obtained will be compared with the experimental measurements. The last section will present the conclusion of this work.

2 Methods

An infinite periodic structure composed of identical patterns is considered. The WFE method reduced the computation of the dynamics of the structure to a wave problem at one boundary of one of its patterns as follows.

In the frequency domain, for every pattern (n), the following equilibrium relationship can be written:

$$\widetilde{\mathbf{D}}\mathbf{q}^{(n)} = \mathbf{F}^{(n)} \tag{1}$$

Where $\mathbf{q}^{(n)}$ contains the nodal displacements of the pattern (n), $\mathbf{F}^{(n)}$ its nodal forces and $\widetilde{\mathbf{D}}$ its stiffness matrix.

Let's note with subscripts I, L and R the components respectively corresponding to the inner, left and right boundaries nodes. Eliminating the displacement of the inner nodes, the reduced equilibrium relationship is found:

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} - \begin{bmatrix} \mathbf{D}_{LI}\mathbf{F}_I \\ \mathbf{D}_{RI}\mathbf{F}_I \end{bmatrix} \tag{2}$$

Where,

$$\begin{array}{ll} \mathbf{D}_{LL} = \widetilde{\mathbf{D}}_{LL} - \widetilde{\mathbf{D}}_{LI}\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IL} & \mathbf{D}_{LR} = \widetilde{\mathbf{D}}_{LR} - \widetilde{\mathbf{D}}_{LI}\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IR} \\ \mathbf{D}_{RL} = \widetilde{\mathbf{D}}_{RL} - \widetilde{\mathbf{D}}_{RI}\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IL} & \mathbf{D}_{RR} = \widetilde{\mathbf{D}}_{RR} - \widetilde{\mathbf{D}}_{RI}\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IR} \\ \mathbf{D}_{LI} = \widetilde{\mathbf{D}}_{LI}\widetilde{\mathbf{D}}_{II}^{-1} & \mathbf{D}_{RI} = \widetilde{\mathbf{D}}_{RI}\widetilde{\mathbf{D}}_{II}^{-1} \end{array} \tag{3}$$

Let $\mathbf{u}^{(n)}$, the vector containing the forces and displacements at the left boundary of the pattern (n), be defined as follows:

$$\mathbf{u}^{(n)} = \begin{bmatrix} \mathbf{q}_L^{(n)} \\ -\mathbf{F}_L^{(n)} \end{bmatrix} \tag{4}$$

Using the continuity of the structure at the pattern boundaries, Hoang *et al.* proved the propagation relationship :

$$\mathbf{u}^{(n+1)} = \mathbf{S}\mathbf{u}^{(n)} + \mathbf{b}^{(n)} \tag{5}$$

Where,

$$\begin{gathered} \mathbf{S} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{LR}^{-1} \\ \mathbf{D}_{RL} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix} \\ \begin{bmatrix} \mathbf{D}_{qI} \\ \mathbf{D}_{fI} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \\ \mathbf{D}_{RI} - \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LI} \end{bmatrix} \\ \mathbf{b}^{(n)} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I^{(n)} \\ \mathbf{D}_{fI}\mathbf{F}_I^{(n)} - \mathbf{F}_{\partial R}^{(n)} \end{bmatrix} \end{gathered} \tag{6}$$

As the vector $\mathbf{u}^{(n)}$ gives the state of the pattern (n), this equation traduces the propagation of the wave from one pattern to the next one. $\mathbf{S}$ can be seen as a propagation matrix and $\mathbf{b}^{(n)}$ gives the effect of the loading applied on the pattern. Then, from the propagation relationship, the following system of equations is obtained. This system reduces the computation of the whole structure dynamics to the computation of $\mathbf{u}^{(0)}$.

$$\begin{cases} \mathbf{u}^{(n)} = \mathbf{S}^n\mathbf{u}^{(0)} + \displaystyle\sum_{k=1}^{n} \mathbf{S}^{n-k}\mathbf{b}^{(k-1)} \\ \mathbf{u}^{(-n)} = \mathbf{S}^{-n}\mathbf{u}^{(0)} - \displaystyle\sum_{k=1}^{n} \mathbf{S}^{-n+k-1}\mathbf{b}^{(-k)} \end{cases} \tag{7}$$

The eigenvalues and eigenvectors $\{\mu_j, \phi_j\}_j$ of $\mathbf{S}$ are used to compute the power of the matrix $\mathbf{S}$. They follow:

$$\mathbf{S}\phi_j = \mu_j\phi_j \tag{8}$$

The computation of this eigenvalue problem is prone to numerical difficulties (see [6]). To overcome them, we use the $\mathbf{S} + \mathbf{S}^{-1}$ transformation proposed by Zhong and Williams [7].

The eigenvalues come in pair $(\mu_j, \mu_j^\star)$ with $\|\mu_j\| \leqslant 1$ and $\mu_j^\star = \frac{1}{\mu_j}$ (see [8]). The corresponding eigenvectors are noted $(\phi_j, \phi_j^\star)$. We define the eigenbasis $\{\mathbf{\Phi}\ \ \mathbf{\Phi}^\star\}$ as: $\mathbf{\Phi} = [\boldsymbol{\phi_1} \dots \quad \boldsymbol{\phi_n}]$ and $\mathbf{\Phi}^\star = [\phi_1^\star \dots \phi_n^\star]$. $\mathbf{\Phi}$ corresponds to the modes propagating to the right and $\mathbf{\Phi}^\star$ to those propagating to the left.

By a condition of non-divergence at infinity, one can show:

$$\mathbf{u}^{(0)} = \mathbf{\Phi} \sum_{k=1}^{+\infty} \boldsymbol{\mu}^{k-1} \mathbf{Q}_E^{(-k)} + \mathbf{\Phi}^\star \sum_{k=0}^{+\infty} \boldsymbol{\mu}^{k+1} \mathbf{Q}_E^{\star(k)} \tag{9}$$

Hoang et al [8] give the formulas to compute the wave amplitudes $\mathbf{Q}_E^{(n)}$, $\mathbf{Q}_E^{\star(n)}$. Computing $\mathbf{u}^{(0)}$ with the last equation gives the dynamics of the whole infinite structure.

3 Results

This section presents some results obtained with the WFE method for a ballastless railway track subjected to a constant moving load. The results are then compared to experimental strain measurements.

One pattern of the structure is modelled in three dimensions using the finite element software Abaqus. The geometry includes the rail, the underlying concrete slab and the support system which connects the rail to the slab. The geometry and mesh obtained are shown in Figure 1. The mesh and the mass, damping and stiffness matrices obtained are exported to Matlab where the WFE computation is conducted. At each frequency, the dynamic stiffness matrix is computed from the mass, damping and stiffness matrices. The structure is subjected to constant moving loads which represent the loads applied by eight wheels moving on the rail at a constant speed.

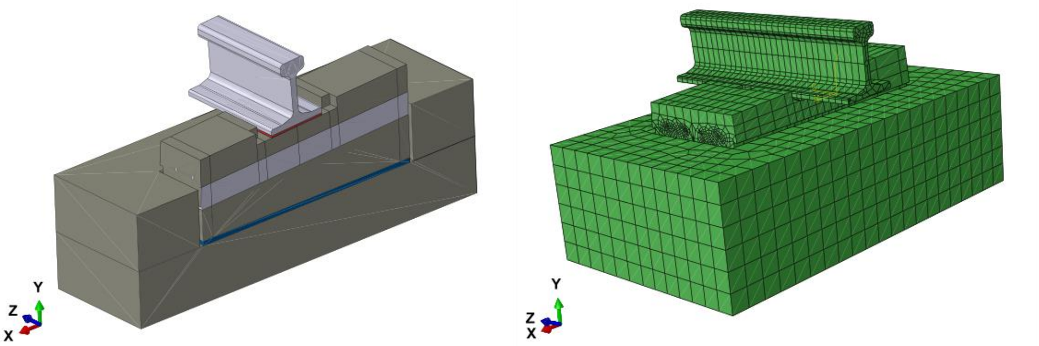

Figure 1: Cross-sectional view of the geometry and mesh of one pattern.

Table 1 gives the parameters used in this computation. The physical parameters correspond to the values measured on the ballastless track of the Channel Tunnel.

Parameter	Value
Steel Young modulus	210 GPa
Steel density	$7.8\,\mathrm{kg/dm^3}$
Under-sleeper pad Young modulus	2 MPa
Under-sleeper pad density	$1\,\mathrm{kg/dm^3}$
Under-rail pad Young modulus	20 MPa
Under-rail pad density	$1\,\mathrm{kg/dm^3}$
Concrete Young modulus	50 GPa
Concrete density	$2.4\,\mathrm{kg/dm^3}$
Support spacing	60 cm
Rail type	UIC60
Sleeper mass	60 kg
Load speed	$38\,\mathrm{km\,s^{-1}}$
Element per pattern	16075
Computation maximum frequency	400 Hz
Computation frequency step	0.66 Hz

Table 1: Computation parameters.

The WFE computation gives the displacements of the nodes of all the patterns in the frequency domain. An inverse Fourier transform is performed to obtain temporal values. Because of Matlab graphical limitations, the three-dimensional results are plotted using Paraview. By a spatial derivation, Paraview is able to compute the strain in the structure. This strain is plotted in Figure 2 at a given time.

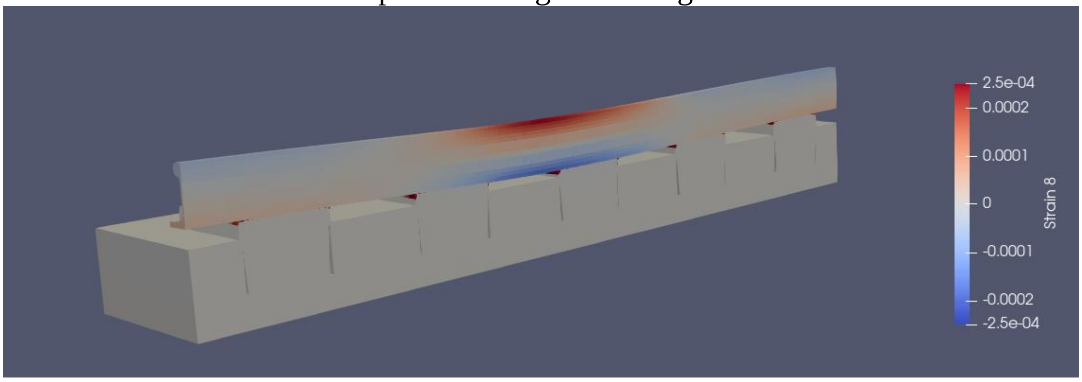

Figure 2: Cross-sectional view of the strain in the track.

To validate these results, a strain gauge was glued on one rail in the Channel Tunnel. In Figure 3, the simulated strain is compared to the measured strain. Although the measured strain shows some experimental noise, a good agreement is found between experimental and numerical values.

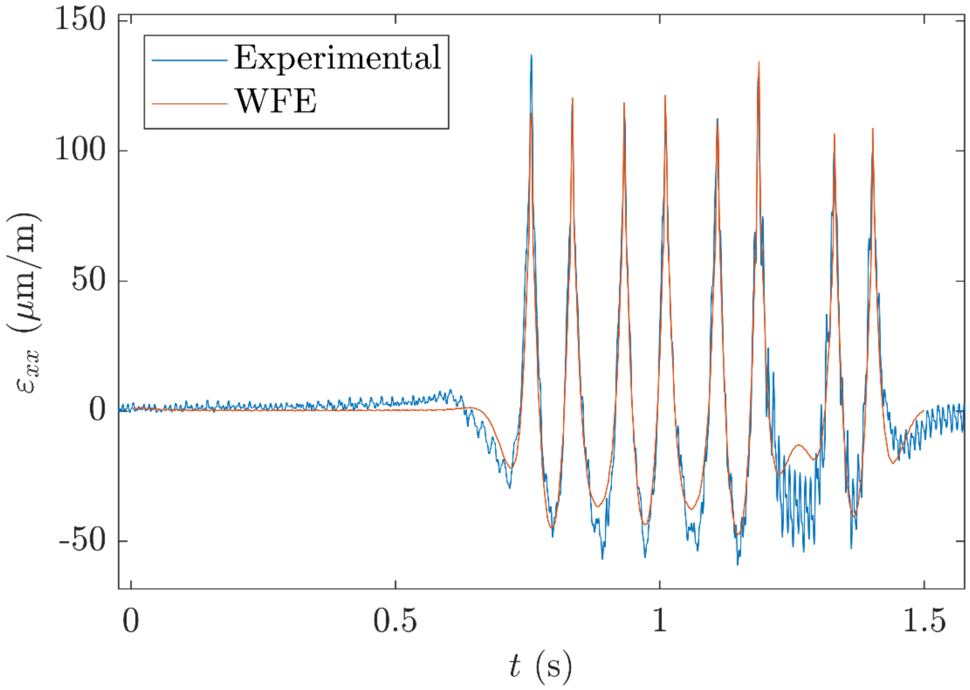

Figure 3: Comparison between computational results and experimental measurements for the longitudinal strain in the rail.

4 Conclusions and Contributions

Railway tracks under traffic can be modelled as infinite periodic structures subjected to moving loads. The Wave Finite Element (WFE) method can be used to compute the dynamics of these structures. This method is based on a reduction of the problem of the computation of the periodic structure dynamic response to a wave problem at a boundary of one of the structure patterns. Intrinsically including the infinite nature of the structure, this reduction can make numerical computations of finely represented complex structures affordable.

The WFE method had already been successfully applied to the railway domain for simple one-dimension models of the track. These models can't give access to stresses and strains at a fine scale. In this article, a three-dimensional fine model is used to represent a ballastless railway track. In this model the rail, the support and the underlying slab are all represented in three dimensions. To validate the results obtained with the Wave Finite Element method, a strain gauge was glued to a rail in the Channel Tunnel. The results obtained show a strong agreement with on-site measurements.

In ongoing studies, the authors are working on this type of computations to model railway transition zones with the same amount of details. An optimization process is also conducted to improve the method efficiency.

Acknowledgements

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