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Convergence Study of Local Hierarchical Functions for Free Vibration Analysis with Application to Multi-Step Beams

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Abstract

The free vibration analysis of homogeneous uniform beams, plates and multi-step beams by the Ritz method, using local Bardell's polynomials and trigonometric functions, is studied in this paper. The first part of the paper presents a comparative study of the convergence of hierarchical sets under both *p*- and *h*-refinements. To this end, a beam and plate are modelled using a single element with a varied number of local functions and multiple elements with a fixed number of local functions. In the second part the above sets of local hierarchical functions are applied for the free vibration analysis of a multi-step beam. With the aim to improve the accuracy of the fundamental mode when using local trigonometric functions, a set of modified local trigonometric functions is proposed to facilitate the satisfaction of the global natural boundary conditions. The use of modified local trigonometric functions with the satisfaction of global natural boundary conditions is shown to significantly improve the accuracy and convergence of the fundamental mode while also converging for higher modes. Moreover, it is also shown to converge under *h*-refinements in contrast to the divergence observed when using standard trigonometric functions.

Keywords: free vibration analysis, Ritz method, multi-step beam, hierarchical functions, *h*-method, *p*-method, Bardell's polynomials, trigonometric functions.

1 Introduction

The standard Rayleigh-Ritz method requires the use of admissible functions, which varies with the problem based on the boundary conditions. The use of hierarchical function set overcomes this limitation and satisfies different essential boundary conditions simply by excluding appropriate functions from the set. Moreover, the hierarchical functions enable a *p*-refinement analysis using the finite element method (FEM) or the Rayleigh-Ritz method, which offers several advantages over the *h*-version like better conditioned matrices and not requiring change of mesh.

Bardell [1] proposed a set of hierarchical polynomials for free vibration analysis of a uniform homogeneous plate. The plate was modelled as a single element and the *p*-version FEA was performed. However, convergence studies under *h*-refinements were not shown. Beslin and Nicolas [2] proposed a hierarchical trigonometric set to accurately predict higher order bending modes as Bardell's polynomials were shown to exhibit significant numerical errors for modes higher than 45. Barrette et al. [3] used this hierarchical trigonometric set for the vibration of stiffened plates. A uniform plate was modelled with a varied number of elements and local functions and they concluded that few discretization elements and high order interpolation functions yield better results for the same degrees of freedom. However, no explicit *h*-refinement and *p*-refinement studies using the trigonometric functions were presented.

Helicopter blades, wind turbine blades, and robotic arms are often modelled as discontinuous beams which require a multi-element model. Hierarchical functions such as Bardell's polynomials and trigonometric functions provide ease of satisfying different boundary conditions with greater versatility and improved rates of convergence, when used locally for a multi-element model, than the global methods.

In this paper, comparisons of convergence for Bardell's polynomials and trigonometric functions under both *h*- and *p*-refinements by the Ritz method are studied using a single and multi-element model of a uniform homogeneous beam and a plate. Next, the local trigonometric functions and Bardell's polynomials are applied for the free vibration analysis of a discontinuous-stepped beam first studied by Jaworski and Dowell [4]. A modification in the trigonometric set is proposed to enable the satisfaction of global natural boundary conditions to improve the accuracy of initial modes when using the trigonometric functions. The results are compared with those obtained by Dang et al. [5] using the trigonometric functions and exact results by Elishakoff et al. [6].

2 Free vibration analysis using local hierarchical functions

Consider a cantilevered Euler-Bernoulli beam divided into various elements along its length with modelled by n local functions per element. The local x coordinate is replaced by a non-dimensional coordinate ξ, as shown in Figure 1. The local displacement field on element (i) is expressed as:

$$w^i(\xi^i, t) = \sum_{j=1}^{n} q_j^{(i)} \cdot \phi_j^{(i)}(\xi^i)$$
$$q_j^{(i)} = \overline{q_j^{(i)}} \cdot e^{i\omega t} \tag{1}$$

For the element containing the fixed end, the first and second functions of the chosen hierarchical set are to be excluded.

Similarly, consider a plate with all edges simply supported, divided into multiple elements with m and n number of local functions along x and y directions respectively. The local x and y coordinates are replaced by ξ and η coordinates respectively, as shown in Figure 1. The local displacement field on element (i) is expressed as:

$$w^i(\xi^i,\eta^i,t) = \sum_{j=1}^{m}\sum_{k=1}^{n} q_{jk}^{(i)} \cdot \phi_j^{(i)}(\xi^i) \cdot \phi_k^{(i)}(\eta^i)$$

$$q_{jk}^{(i)} = \overline{q_{jk}^{(i)}} \cdot e^{i\omega t} \tag{2}$$

For the element containing a simply supported edge, the first function of the chosen hierarchical set is to be excluded. Using Euler-Lagrange's equation, local equations of motion are obtained. The global assembly of the local equations followed by the condensation procedure yields the final system of linear equations.

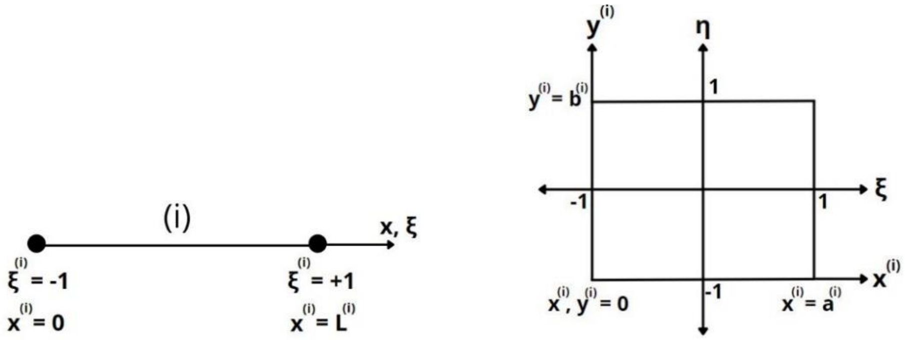

Figure 1: Non-dimensional coordinates for element (i) of a beam and a plate.

Here, we propose a modification in the hierarchical trigonometric set which is used to satisfy the global natural boundary conditions for improving the accuracy of initial modes when using trigonometric functions. The modified set ψ_j is defined as:

$$\psi_j(\xi) = \phi_j(\xi) + p_j(\xi)$$

$$p_j(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + c_6\xi^6 + c_7\xi^7 \tag{3}$$

for $1 \leq j \leq 8$. ϕ_j are the Beslin and Nicolas' [2] trigonometric functions.

The eight unknown coefficients in p_j are determined by satisfying eight boundary conditions on ψ_j as shown in Table 1. For $j > 8$,

$$\psi_j(\xi) = \phi_j(\xi) \times (1 - \xi^2)^2 \tag{4}$$

Now, the free end natural boundary conditions for a cantilevered beam can be satisfied by excluding ψ_5 and ψ_6 when defining the local displacement field over the element containing the free end.

	$\xi = -1$				$\xi = +1$			
j	ψ_j	$\psi_j{}'$	$\psi_j{}''$	$\psi_j{}'''$	ψ_j	$\psi_j{}'$	$\psi_j{}''$	$\psi_j{}'''$
1	1	0	0	0	0	0	0	0
2	0	$\pi/2$	0	0	0	0	0	0
3	0	0	0	0	1	0	0	0
4	0	0	0	0	0	$\pi/2$	0	0
5	0	0	0	0	0	0	1	0
6	0	0	0	0	0	0	0	1
7	0	0	1	0	0	0	0	0
8	0	0	0	1	0	0	0	0

Table 1: Boundary conditions imposed on ψ_1 to ψ_8.

3 Results

The cantilevered uniform beam of dimensions $25.4 \times 3.175 \times 463.55$ *mm*, is analysed for out-of-plane bending deflections. The Young's modulus and density (ρ) are 60.6 GPa and 2664 Kg/m^3 respectively. For the corresponding uniform plate *b/a* = 1 and Poisson's ratio (ν) is 0.3. For the plate, a non-dimensional frequency Ω is defined as:

$$\Omega = \sqrt{\frac{\rho h a^4}{D}}\ \omega \qquad (5)$$

3.1 Convergence comparison under *p*-refinements

The uniform beam and plate are modelled as a single element with the number of local functions increased sequentially. The results are shown in Figures 2-5 for the fundamental mode and a representative higher mode.

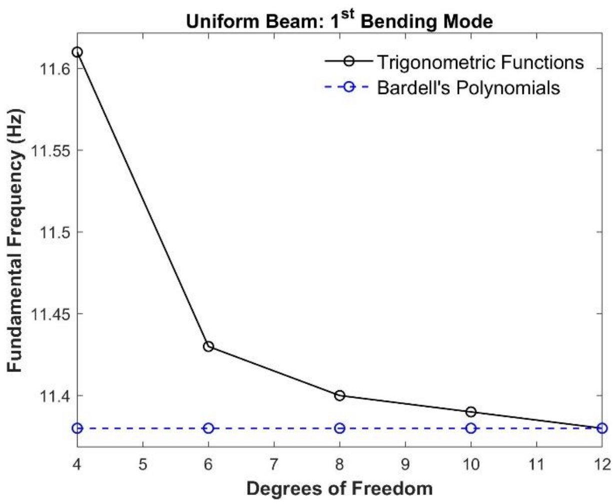

Figure 2: Convergence under *p*-refinements for uniform beam: Fundamental mode.

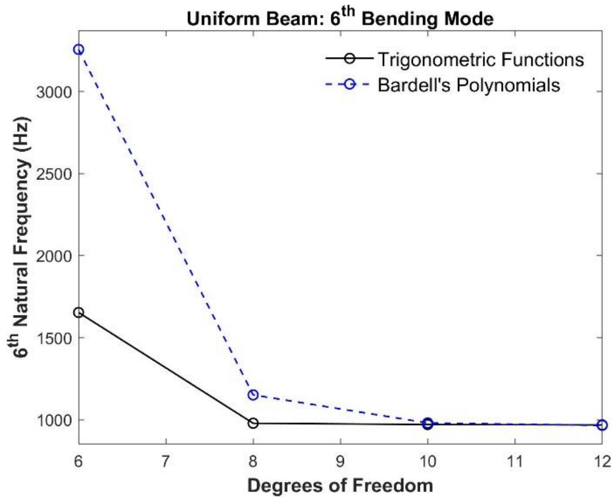

Figure 3: Convergence under *p*-refinements for uniform beam: 6th mode.

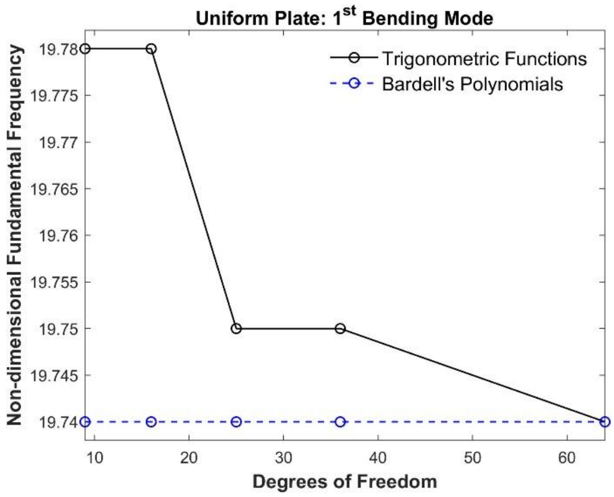

Figure 4: Convergence under *p*-refinements for uniform plate: Fundamental mode.

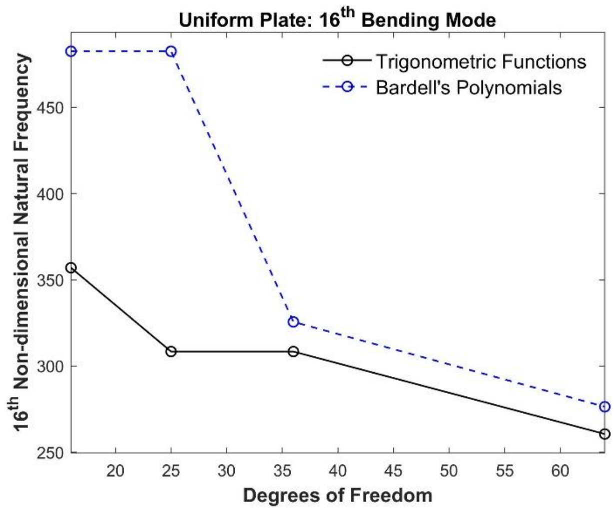

Figure 5: Convergence under *p*-refinements for uniform plate: 16th mode.

3.2 Convergence comparison under *h*-refinements

In this analysis, fixed number of local functions are used with the number of elements increased sequentially. The results are shown in Figures 6-9.

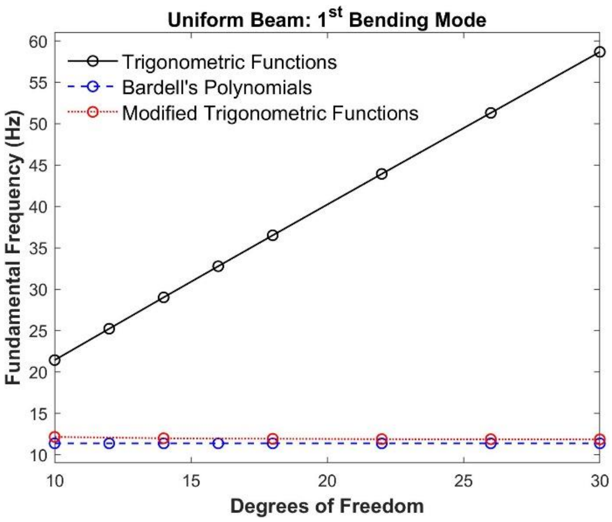

Figure 6: Convergence under *h*-refinements for uniform beam: Fundamental mode.

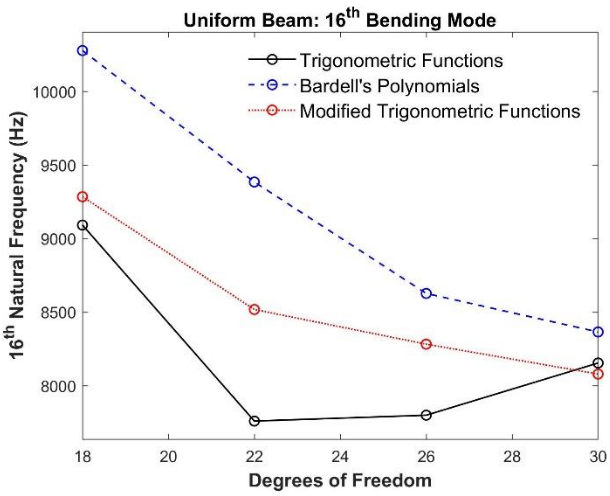

Figure 7: Convergence under *h*-refinements for uniform beam: 16^{th} mode.

3.3 Comparison of hierarchical functions applied for a discontinuous beam

The multi-step beam [4] shown in Figure 10, is analysed. Each of the 13 uniform domains is modelled as one element with a varied number of local functions. The results are shown in Figures 11 and 12. Table 2 shows the comparison of the present results with those obtained by Dang et al. [5], with the error calculated using the exact result obtained by Elishakoff et al. [6].

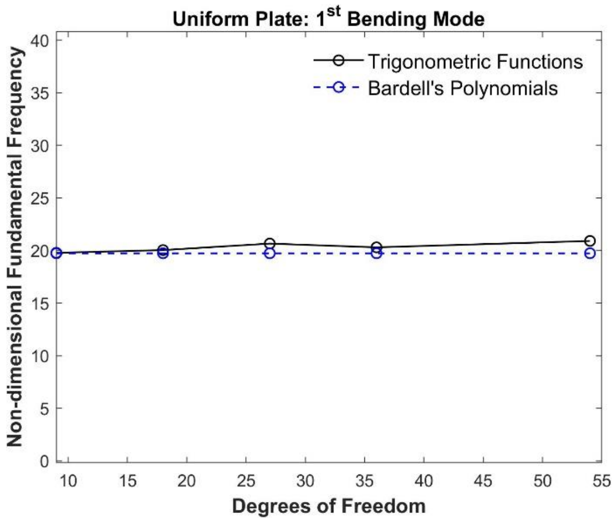

Figure 8: Convergence under *h*-refinements for uniform plate: Fundamental mode.

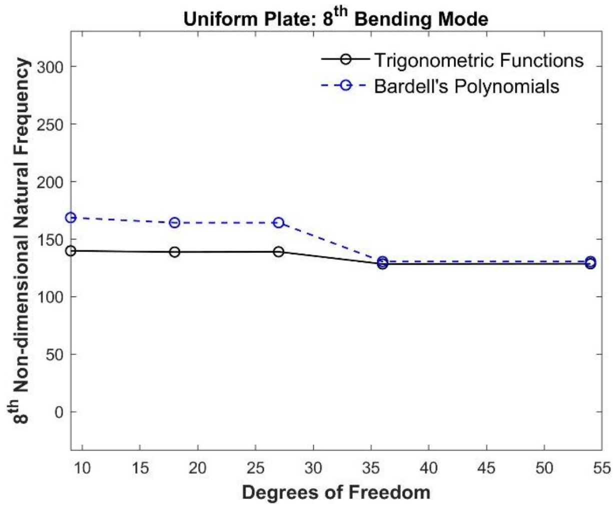

Figure 9: Convergence under *h*-refinements for uniform plate: 8^{th} mode.

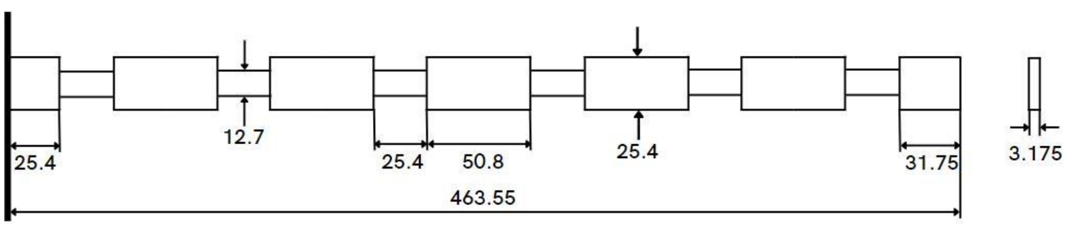

Figure 10: Multi-step cantilevered Euler-Bernoulli beam.

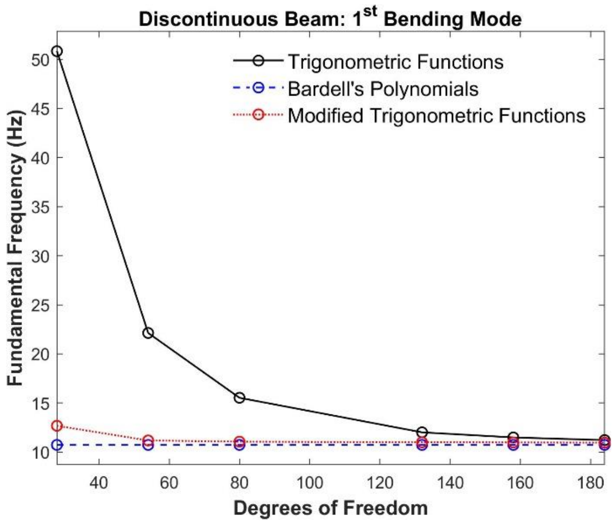

Figure 11: Convergence under *p*-refinements for stepped beam: Fundamental mode.

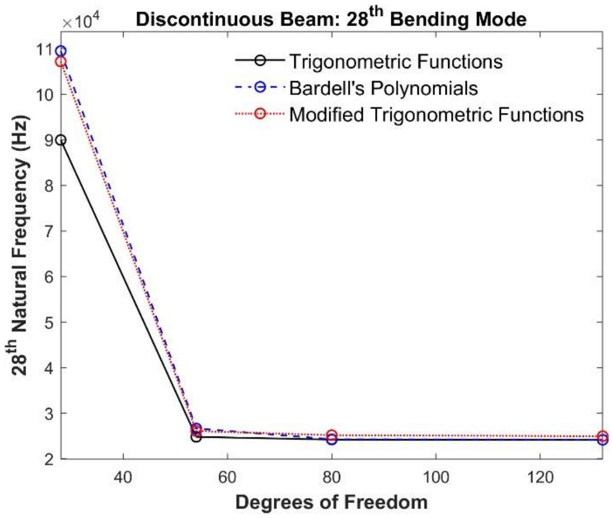

Figure 12: Convergence under *p*-refinements for stepped beam: 28^{th} mode.

A monotonic convergence is observed for all sets under *p*-refinements. Divergence for trigonometric functions, whereas convergence for others is observed under *h*-refinements.

	Dang et al. [5]		Present (Trigonometric functions)		Present (Modified trigonometric functions)	
Local functions	ω_1 [Hz]	Error [%]	ω_1 [Hz]	Error [%]	ω_1 [Hz]	Error [%]
8	10.761	0.147	15.536	44.6	11.194	4.18
12	10.743	-0.019	12.016	11.8	11.007	2.44
18	-	-	11.070	3.02	10.952	1.93

Table 2: Comparison of results for the fundamental frequency (ω_{exact} = 10.7451 Hz).

4 Observations and Conclusions

In this paper, three sets of local hierarchical functions, namely Bardell's polynomials [1], trigonometric functions [2], and modified trigonometric functions to satisfy the natural boundary conditions, are studied for free vibration analysis by the Ritz method. Initially, a uniform homogeneous beam and plate are modelled using single and multi-elements. Following this, the hierarchical sets are applied for the free vibration analysis of a multi-step beam. From the *p*- and *h*-refinement study of various configurations, the following observations can be drawn:

- Under the *p*-refinement study, all sets show a monotonic convergence for all the modes. For the initial modes, Bardell's polynomials yield better convergence, whereas, for higher modes, trigonometric functions result in better convergence. As shown in Figures 3 and 5, even for modes much lower than 45, for which the numerical errors are not yet significant in Bardell's polynomials, the trigonometric functions are observed to yield better convergence and accuracy.
- Under the *h*-refinement study, Bardell's polynomials and modified trigonometric functions show a monotonic convergence, whereas trigonometric functions show divergence for the fundamental mode and initial convergence followed by divergence for higher modes. The divergence with increase in degrees of freedom is less significant in the plate compared to the beam. The modified trigonometric functions, when compared with Bardell's polynomials, are observed to yield better convergence and accuracy for higher modes and comparable results for lower modes.
- For lower modes, Bardell's polynomials result in the best convergence and accuracy, whereas the trigonometric functions yield poor results and slower convergence, which is significantly improved by the satisfaction of natural boundary conditions using the modified trigonometric functions.
- For higher modes, trigonometric functions result in the best convergence and accuracy with the modified trigonometric functions also converging to results of comparable accuracy.
- From Table 2, it is noted that the use of trigonometric functions does not yield results obtained by Dang et al. [5]. Although in both the studies, the convergence begins from an upper bound, the frequency obtained by Dang et al. is found to go below the exact value. The modified trigonometric functions together with satisfying the natural boundary conditions show superior accuracy and convergence than trigonometric functions.

From this study, it is concluded that the modified trigonometric functions together with satisfying the natural boundary conditions provide excellent convergence and accuracy for all the modes in both uniform and multi-step beams.

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