The Fourteenth International Conference on Computational Structures Technology
23–25 August, 2022 | Montpellier, France

Proceedings of the Fourteenth International Conference on
Computational Structures Technology
Edited by B.H.V. Topping and J. Kruis
Civil-Comp Conferences, Volume 3, Paper 14.2
Civil-Comp Press, Edinburgh, United Kingdom, 2022, doi: 10.4203/ccc.3.14.2

Analytical Elasticity Solution for the Stress Analysis of Arbitrarily-supported Isotropic Beams under Localized Loads

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Abstract

A two-dimensional elasticity-based analytical solution is presented for accurate stress analysis of arbitrarily-supported isotropic beams subject to patch loads. The 2D elasticity-based system of governing equations is formulated in mixed form by employing the Reissner-type variational principle and solved analytically for arbitrary boundary conditions by employing a multi-term extended Kantorovich (EKM) approach. The correctness and efficacy of the present mechanical model are established by comparing its results with finite element analysis solutions. It is shown that the analytical formulation is capable of capturing the displacement and highly localized stress concentration profile arising from patch loads both in thin and thick geometric configurations, outperforming classical laminate theory predictions.

Keywords: elasticity solution, localized stresses, discontinuous load, patch load

1 Introduction

Metallic beams are essential structural components that are widely used in engineering practice. The computation analysis of their structural response requires dedicated, accurate models. In recent years, many mathematical models have been developed for static, vibration, and buckling investigation of beams [1–3]. In the context of stress analysis of laminated beams, Lekhnitskii [4] introduced Airy-stress polynomial functions for exact elasticity analysis of beams. Using these Airy stress polynomial functions approach, Silverman [5], Hasin [6], Gerstner [7], Rao and Ghosh [8] and Cheng et al. [9] developed exact solutions for laminated beams. An elasticity-based analytical solution for the bending analysis of beam structures is developed by Esendemir et al. [10].

In the elasticity framework, the development of the analytical solution for arbitrary boundary conditions is very challenging due to computational difficulties and the complexity of laminated systems. Therefore, many researchers also explored pure numerical approaches for beams [3]. Elasticity based semi-analytical solution was developed by Chen et al. [11] for flexural and natural frequency analysis of the laminated beams using the differential quadrature method (DQM). Recently, Subramanian and Mulay [12] extended Pagano's theory for the flexural analysis of laminated beams. Trinh et al. [13] utilized the inverse differential quadrature method with zigzag theory and Doeva [14] employed a Variational Iteration Method (VIM) for the static analysis of laminated composite beams.

The aforementioned analytical and numerical solutions are limited to uniformly distributed continuous loads. In the context of localized loading, Pagano [15] developed an exact solution for simply-supported laminates under cylindrical bending and subjected to concentrated and distributed loads. Later, Kapuria et al. [16] assessed the zigzag theory under plane-stress conditions for simply-supported composite beam subjected to static patch load. To the author's best knowledge, no analytical solution has been reported in the literature for arbitrarily supported beams subjected to patch loads. However, discontinuous loads are highly common in practice. As a result, the capability of the mechanical analysis models to accurately and rigorously assess the internal stress state of such structural components to general types of loading and support conditions is of primal importance.

In this article, we aim at proposing an accurate 2D analytical solution for the static analysis of arbitrarily supported isotropic beams subjected to locally distributed loads for the first time. To that scope, we extend the Kantorovich method in the discrete loading analysis space.

2 2D elasticity-based formulation for isotropic beams

A single-layer isotropic beam (x = (0, a), z = $-h/2$, $h/2$), as presented in Figure 1, is considered for the present study. Here. $\zeta = \left(z-h/2\right)/t$ is a non-dimensionalized thickness

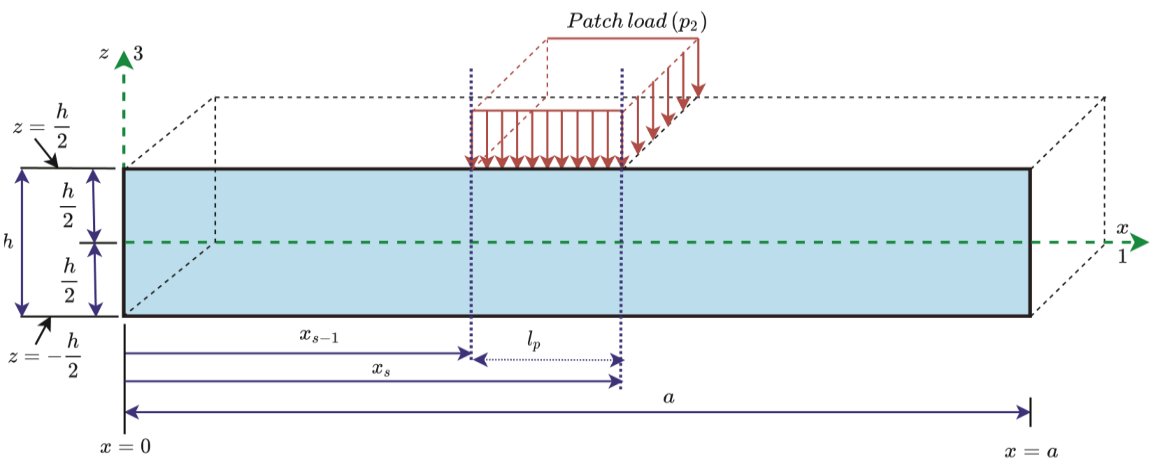

Figure 1: Geometry of the isotropic beam considered for the present study.

parameter and t denotes the thickness of the beam. Similarly, the beam span is divided into segments with and without loading and $\xi^{(s)} = (x_s - x_{s-1})/l^{(s)}$ is a non-dimensional axial parameter defined for each segment, with $l^{(s)}$ denoting the length of that segment. The superscript 's' will be omitted in the further mathematical expressions unless specifically used for clarity. The parameter $\xi_1 = x/a$ is a global in-plane parameter.

The linear strain-displacement relations for the 2D straight beam can be written as, $\varepsilon_x = u,_x$; $\gamma_{zx} = w,_x + u,_z$ and $\varepsilon_z = w,_z$. The elasticity-based constitutive equations for the isotropic beam can be expressed as, $\varepsilon_x = s_{11}\sigma_x + s_{13}\sigma_z$; $\varepsilon_z = s_{13}\sigma_x + s_{33}\sigma_z$; $\gamma_{zx} = s_{55}\tau_{zx}$. The elastic compliances s_{ij} are given by, $s_{11} = s_{33} = 1/E$; $S_{55} = 1/G$; $S_{13} = -\nu/E$. Here, *E*, *G* and *v* denote Young's modulus, shear modulus and Poisson's ratio of the isotropic beam, respectively.

Using strain-displacement and constitutive equations, Reissner-type mixed variational principle for a isotropic beam without body force can be written as,

$$\int_V \delta\sigma_x\{s_{11}\sigma_x + s_{13}\sigma_z - u,_x\} + \delta\sigma_z\{s_{13}\sigma_x + s_{33}\sigma_z - w,_z\} + \delta\tau_{zx}\{s_{55}\tau_{zx} - u,_z - w,_x\}] + \delta u(\sigma_{x,x} + \tau_{xz,z}) + \delta w(\tau_{zx,x} + \sigma_{z,z})dV = 0 \quad (1)$$

It is assumed that the top surface and bottom surface of the beam are shear tractions free ($\tau_{zx} = 0$) and the beam is subject to a uniform distributed patch load p_2 at the top surface. The displacements (*u* , *w*) and transverse stresses (σ_z, τ_{zx}) at the segment interfaces need to satisfy the following condition at the inter-segment interface

$$[(u,\ w,\ \sigma_x,\ \tau_{zx})|_{\xi^s=1}]^{(s)} = [(u,\ w,\ \sigma_x,\ \tau_{zx})|_{\xi^s=0}]^{(s+1)} \quad (2)$$

Along *x*-axis isotropic beam can have any type of support such as, Simply-supported: $\sigma_x = w = 0$; Clamped: $u = w = 0$; and Free: $\sigma_x = \tau_{xz} = 0$.

3 EKM analytical solution approach

In the present mathematical model, both displacements (*u*, *w*) and stresses ($\sigma_x, \sigma_z, \tau_{zx}$) are considered as primary variables and solved using the multi-term extended Kantorovich method [17,18]. The field variables for a segment can be expressed as follows:

$$[u\ \ w\ \ \sigma_x\ \ \sigma_z\ \ \tau_{zx}]^T = \sum_{i=1}^{n}[f_1^i g_1^i\ \ f_2^i g_2^i\ \ f_3^i g_3^i\ \ f_4^i g_4^i + [p_a^s + z p_d^s]\ \ f_5^i g_5^i\ ]^T \quad (3)$$

where f_l and g_l are unknown functions of ξ and ζ for the *x* and *z*-direction, respectively, and $p_a^s = -p_2^s/2$ and $p_d^s = -p_2^s/h$. The functions $f_l^i(\xi)$ depend on the s_{th} segment while the $g_l^i(\xi)$ functions are valid for all segments.

3.1 First iterative step - solving functions $g_l^i(\zeta)$

In this iteration step, the thickness function $g_l^i(\zeta)$ will be solved and obtained in closed-form. Hence, variation can be expressed as follows:

$$[\delta u\ \ \delta w\ \ \delta\sigma_x\ \ \delta\sigma_z\ \ \delta\tau_{zx}]^T = \sum_{i=1}^{n}[f_1^i\delta g_1^i\ \ f_2^i\delta g_2^i\ \ f_3^i\delta g_3^i\ \ f_4^i\delta g_4^i\ \ f_5^i\delta g_5^i\ ]^T \quad (4)$$

The functions $g_l^i(\zeta)$ can be divided into two parts as $\bar{G}$ and $\hat{G}$. Here $\bar{G}$ is a column vector of dimension 4*n* and contains independent variables that appear in the boundary conditions of the top and bottom surfaces of the beam, while $\hat{G}$ is a column vector of size 1*n* and contains the remaining dependent variables. Considering the solution Equation (3), and its variational part Equation (4) and substituting into Equation (2), integration by parts along the *x*-axis can be performed. Since the variation is arbitrary, the coefficient of δg_l^i must vanish, which yields the following set of governing equations for $\bar{G}$ and $\hat{G}$, as follows:

$$\bar{\mathbf{G}}_{,\zeta} = \mathbf{M}^{-1}\left[\bar{\mathbf{A}}\bar{\mathbf{G}} + \hat{\mathbf{A}}\hat{\mathbf{G}} + \bar{\mathbf{Q}}_{\mathbf{p}}\right] \quad (5)$$

$$\hat{\mathbf{G}} = \left[\mathbf{K}^{\mathbf{m}}\right]^{-1}\left[\tilde{\mathbf{A}}\bar{\mathbf{G}} + \tilde{\mathbf{Q}}_{\mathbf{p}}\right] \quad (6)$$

Here, $\mathbf{M}, \bar{\mathbf{A}}, \hat{\mathbf{A}}, \mathbf{K}$, and $\tilde{\mathbf{A}}$, are coefficient matrices. The substitution of $\hat{G}$ from Equation (6) into Equation (5) transforms Equation (5) as follows:

$$\bar{\mathbf{G}}_{,\zeta} = \mathbf{A}\bar{\mathbf{G}} + \mathbf{Q}_{\mathbf{p}} \quad (7)$$

where Equation (7) is a set of 4*n* first-order coupled ODEs with constant coefficients and this system of ODEs can be solved analytically by following the solution approach suggested by Kapuria and Kumari [19].

3.2 Second iterative step - solving functions $f_l^i(\xi_1)$

In first iterative step $g_l^i(\zeta)$ functions have been obtained in closed-form manners. Now, in this iteration step, the obtained $g_l^i(\zeta)$ functions are used to identify the f_l^i functions, which are assumed as unknown in this iteration. Therefore, variation is assumed in the f_l^i functions as follows:

$$[\delta u \quad \delta w \quad \delta\sigma_x \quad \delta\sigma_z \quad \delta\tau_{zx}]^T = \sum_{i=1}^{n}[g_1^i\delta f_1^i \quad g_2^i\delta f_2^i \quad g_3^i\delta f_3^i \quad g_4^i\delta f_4^i \quad g_5^i\delta f_5^i\ ]^T \quad (8)$$

Similar to the first iteration, the in-plane functions $f_l^i(\xi)$ can also be split up into two column vectors $\bar{F}$ and $\hat{F}$. Here, $\bar{F}$ contains these specific independent variables that appear in the inter-segment continuity and edge conditions along the *x*-directions and the $\hat{F}$ column vector contains the remaining dependent variables. Substituting Equation (8) in Equation (2) and equating the coefficient of δf_l^i to zero individually after performing integration along the *z*-direction leads to a set of differential-algebraic equations as follows:

$$\bar{\mathbf{F}}_{,\xi} = \mathbf{N}^{-1}\left[\bar{\mathbf{B}}^{\mathbf{f}}\bar{\mathbf{F}} + \hat{\mathbf{B}}^{\mathbf{f}}\hat{\mathbf{F}} + \bar{\mathbf{P}}_{\mathbf{m}}^{\mathbf{f}}\right] \quad (9)$$

$$\hat{\mathbf{F}} = \mathbf{L}^{-1}\left[\tilde{\mathbf{B}}^{\mathbf{f}}\bar{\mathbf{F}} + \tilde{\mathbf{P}}_{\mathbf{m}}^{\mathbf{f}}\right] \quad (10)$$

However, $g_l^i(\zeta)$ are already known in closed-form manners from the previous iteration. Hence above set of equations can be solved similarly by applying the edge conditions and intersegment continuity conditions along the *x*-direction on the $\bar{F}$. These two iterative steps can be continued to get the final converged solution up to the needed level of accuracy.

4 Numerical results and discussion

In this section, the effect of localized loading on the flexural response of isotropic beams is investigated. A single-layer aluminum (E=70Gpa, ν=0.3) beam, as shown in Figure 1, is considered for the current numerical study. Numerical results are presented for various boundary conditions and thickness ratios ($S = a/h$). The obtained results are expressed in non-dimensional form, normalized as follows

$$(\bar{u},\bar{w}) = 100(u, w/S)E_0/p_0hS^3;\ (\bar{\sigma}_x,\ \bar{\tau}_{zx}) = (\sigma_x,\ S\tau_{zx})/p_0S^2 \quad \text{with } E_0 = 70GPa.$$

To explore the longitudinal distribution of deflections and stresses under localized patch loads, the variation of deflections and stresses are depicted in Figure 2 for an isotropic beam subject to a locally distributed patch load of length 0.1a at its center. The numerical results

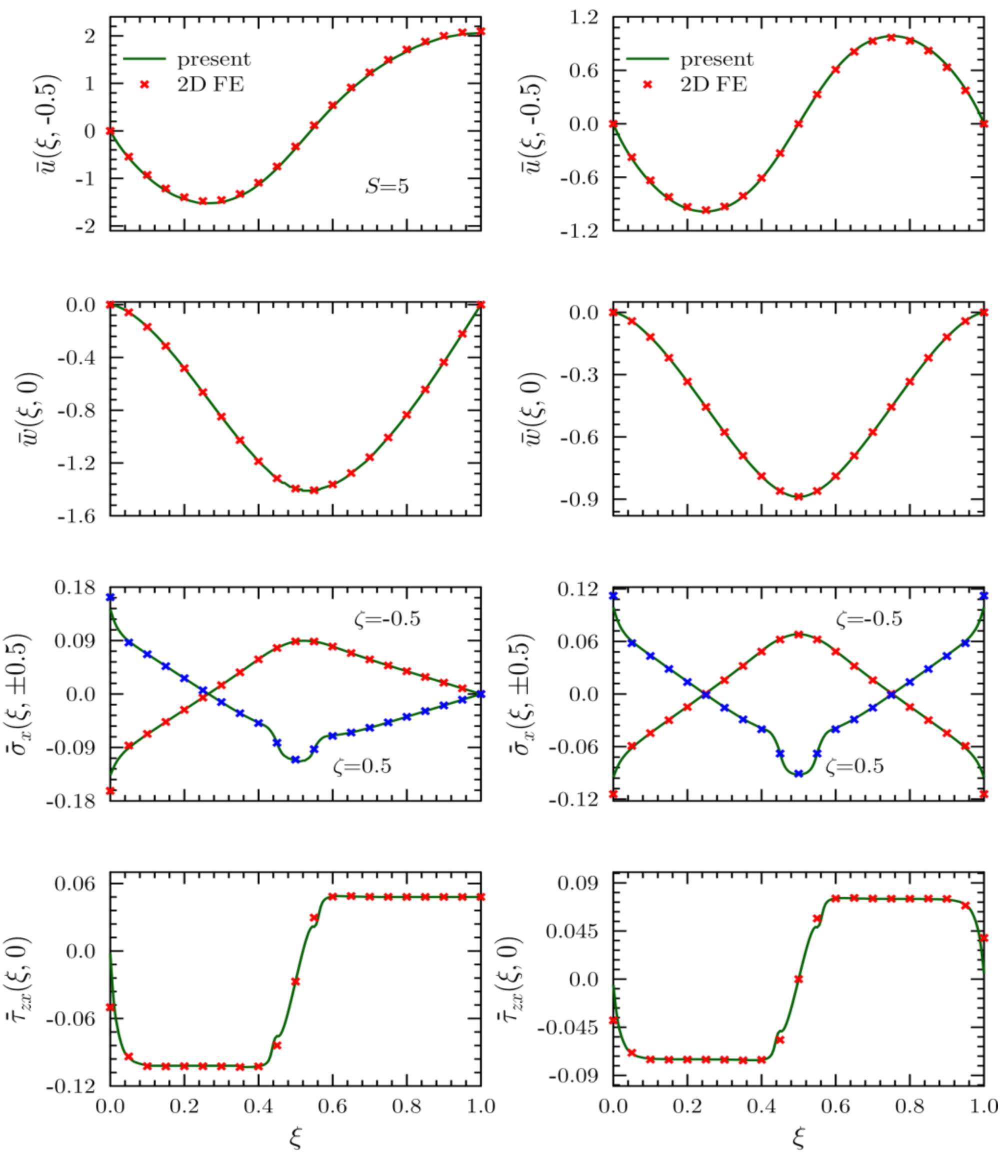

Figure 2: Longitudinal variations of displacements and stresses under S-S and C-C boundary conditions for thick (S=5) isotropic beam subjected to the locally distributed patch load of length 0.1a at center.

are plotted for clamped–simply supported (C-S) and clamped–clamped supported (C-C) end conditions. For validation purposes, 2D FE results are also depicted in Figure 2, along with the present analytical results. Excellent agreement is observed between the two solutions. It is noted that the locally distributed load leads to very high localized normal stress concentration in the vicinity of the applied load, while the stress concentration at the bottom of the beam is significantly lower compared to the top. It is also observed that in the vicinity of the localized distributed load, the transverse stress ($\bar{\tau}_{zx}$) changes from negative to positive in a sharp manner. The present analytical solution can predict the sharp variation of axial and transverse stresses in the vicinity of localized load accurately and efficiently. Moreover, the variation of deflections is smooth along the length of the beam as compared to stresses. Further insights in the stress distribution of the beam upon the application of highly concentrated patch loads are presented in 2D contour form in Figures 3

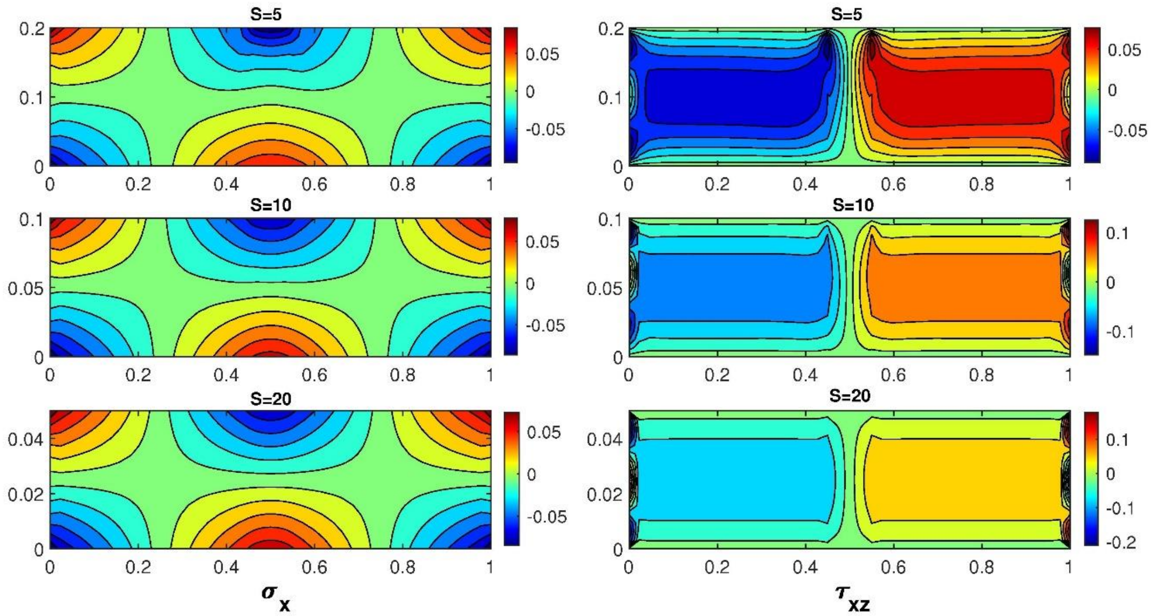

Figure 3: Variation of normal stress σ_x and shear stress τ_{zx} for C–C isotropic beam with different thickness ratios, S = 5, 10, and 20 subjected to a patch load of length 0.1l at center.

for the $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ stress fields in the C–C case. The variation of normal stresses $\bar{\sigma}_x$ is highly localized in the thick beam. In the thick (S =5) and moderately thick (S =10) case, high-stress concentration appear at the top surface in the vicinity of the localized load, and the distribution of normal and shear stresses is highly asymmetric across the thickness. However, for a thin (S =20) geometry, the stress distribution is highly distributed, smooth, and symmetric, as it would be predicted with the use of classical beam theories.

5 Conclusions

A two-dimensional analytical solution has been developed for the accurate stress analysis of arbitrarily-supported isotropic beams subject to distributed patch loads. The Reissner-type mixed variational principle-based formulation has been developed in mixed form. The extended Kantorovich method has been employed to obtain the analytical solution for arbitrary support conditions. An extensive numerical study has been performed to assess the effect of patch loads on the flexural behavior of isotropic beams. It has been found that the present model can efficiently and accurately capture the flexural response under patch loading. The highly localized stress concentration in the vicinity of the applied load are accurately computed, along with their variation in the vicinity of the patch load, which is highly asymmetric and nonlinear along the beam thickness. The analytical results provided in this paper are expected to serve as a benchmark for the general-case, analytical evaluation of the static bending response of beams-structures subject to discontinuous loads, beyond the applicability limits of classical laminate theories.

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