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Relative probabilistic entropies in engineering reliability analysis

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Abstract

This work aims an idea of application of various relative entropies models in reliability analysis of the most popular civil engineering steel structures. Reliability indices computed according to the Cornell theory, included in the Eurocode 0 approach, have been contrasted here with these coming from Kullback-Leibler, Bhattacharyya, Jensen-Shannon as well as Mahalanobis. It was necessary for this purpose to calculate the first two probabilistic moments of different structural responses like extreme stresses and/or deformations assuming Gaussian distributions of both design parameters and structural output. Numerical analysis of this moments has been completed with the use of triple Stochastic Finite Element Method. It was implemented according to (*i*) the iterative generalized stochastic perturbation technique, (*ii*) Monte-Carlo simulation as well as (*iii*) the semi-analytical integral approach. This variety of probabilistic methods allows us to recognize the applicability and accuracy of various stochastic algorithms in addition to the input uncertainty level, to a number of random trials in statistical approach, and to order of Taylor expansion in the stochastic perturbation approach. Finally, the resulting relative entropies have been all rescaled to the variability interval of the First Order Reliability Method.

Keywords: probabilistic entropy, relative entropy, Stochastic Finite Element Method, structural analysis, reliability assessment.

1 Introduction

Relative probabilistic entropy called also probabilistic divergence [9] has been introduced to have a universal measure of a distance in-between probability distributions. It has been defined using various concepts in probability theory and statistics [4] close to maximum entropy principle [8,11], which could be expressed via closed-form analytical equations for more popular probability distributions like Gaussian or lognormal, for instance, assuming that both distributions are of exactly the same type. It seems that relative entropy models meet very close the needs of reliability analysis and show almost the same parameter sensitivity, where admissible state functions and their extreme counterparts most frequently have both statistical nature. Let us note that entropy propagation in some structural problems has been verified before in the literature devoted to stochastic computational dynamics [2].

This work aims at the application of various relative entropies models in the reliability analysis of the most popular civil engineering thin-walled structures. The first two probabilistic moments of different structural responses are determined in the few case studies with the use of the Stochastic Finite Element Method (SFEM) [6] implemented according to three different probabilistic methods – the iterative generalized stochastic perturbation technique, Monte-Carlo simulation as well as the semi-analytical approach. This variety of probabilistic methods allows us to recognize the applicability and accuracy of stochastic analysis in addition to the input uncertainty level, to a number of random trials in statistical approach, and to order of Taylor expansion in the stochastic perturbation approach. These methods have been implemented all in the computer algebra system MAPLE and used with the FEM programs Autodesk ROBOT and ABAQUS. Numerical illustrations include static linear and nonlinear, and also the dynamic response of such steel and aluminum structures as classical simply supported steel plate girder, plane truss girder as well as spatial steel hall structure; the cylindrical metal bars under tensile failure is also considered. The FEM discretizations have been completed in this work using 2 & 3D two-noded bar elements, 3D hexahedral and/or tetrahedral finite elements with linear and quadratic shape functions, and also 3D beam elements with 7 degrees of freedom.

Finally, the reliability indices computed according to the Cornell theory [6], included in the Eurocode 0 approach, have been contrasted here with the Kullback-Leibler [7], Bhattacharyya [3], Jensen-Shannon [5] as well as Mahalanobis [10] relative entropies to check the applicability of these entropies in modern reliability analysis. This has been done for some Gaussian input structural uncertainties representing geometric or material imperfections as well as some statistical dispersions in the external loadings. Numerical analysis has been based upon three different probabilistic methods mentioned above implemented with both well-known analytical elasticity solutions as well as the Least Squares Method approximations following the series of the FEM experiments. It is demonstrated that relative entropy-based reliability analysis could be further extended towards time-dependent engineering structures and systems accounting for corrosion or some other form of stochastic aging.

2 Methods

The Stochastic Finite Element Method is the fundamental computational tool in this work to assess the reliability of some civil engineering structures with the use of relative entropies. It has been based upon numerical recovery of polynomial bases linking specific input uncertainty sources with the state structural functions. These bases have been completed using some series of the FEM experiments and the Least Squares Method in the following form convenient for some discrete time moment τ:

$$u_j(\tau;b) \cong \sum_{i=1}^{n} A_{ji}(\tau) b^i \,. \tag{1}$$

Polynomial bases under interest have been further used in a triple probabilistic analysis – using (*i*) statistical estimation of the Monte-Carlo simulation [1,5], (*ii*) higher-order Taylor expansion typical for the generalized iterative stochastic perturbation method (of the tenth order), and also (*iii*) probabilistic integral semi-analytical method [6]. Such a triple approach has been entirely programmed in the computer algebra system MAPLE 2019. Some structural geometrical and material imperfections have been assumed as the Gaussian random parameters with the given expectation and some small interval for input coefficient of variation fluctuation (its upper limit has been set as equal to 0.20). Thanks to the common implementation of ABAQUS and MAPLE the first four probabilistic characteristics, i.e. expectations, coefficients of variation, skewness, and kurtosis of extreme structural deformations and stresses have been determined. Expected values are calculated, for example, from three different formulas corresponding to the aforementioned methods as

$$E\left[u_j(\tau;b)\right] = \int_{-\infty}^{+\infty} \sum_{i=1}^{n} A_{ji}(\tau) b^i \; p_b(x) dx \,, \tag{2}$$

$$E\left[u_j(\tau;b)\right] \cong \frac{1}{M} \sum_{m=1}^{M} \sum_{i=1}^{n} A_{ji}(\tau) b_m^i \,, \tag{3}$$

$$E\left[u_j(\tau;b)\right] \cong \int_{-\infty}^{+\infty} \left\{ u_j^0(\tau;b^0) + \sum_{i=1}^{N} \varepsilon^i \frac{\partial^i u_j(\tau;b^0)}{\partial b^i (b=b_0)} \left(b-b^0\right)^i \right\} p_b(x) dx \,. \tag{4}$$

M stands for the total number of random trials in the MCS procedure, N denotes stochastic perturbation method order in SPT technique, the upper script ‘0’ denotes mean value of the given parameter, whereas ε is the perturbation parameter.

Further, these methods have been compared with each other to detect by the way applicability limits for the stochastic perturbation technique, which is the least time and computer power-consuming approach here. Such a numerical strategy guarantees any desired numerical accuracy in the FEM discretization for even multiscale engineering structures, common usage of any finite elements, and approximating functions available in the ABAQUS with relatively high precision of the stochastic modeling itself.

The second part of this study has been devoted to the determination of the reliability indices. The First Order Reliability Method included in the designing code Eurocode 0 has been compared with various theories describing relative probabilistic entropy, namely Bhattacharyya, Kullback-Leibler, and also some other. This part of computer analysis has been entirely delivered in the computer algebra system MAPLE with the use of the previously computed first two probabilistic moments of the desired structural response. A comparison of various reliability indices resulting from the FORM and these models has been provided separately for the three aforementioned stochastic computer methods. This comparison has been additionally parametrized with the input coefficient of variation to study possible convergence (or divergence) of the numerical apparatus proposed with increasing uncertainty in the models.

3 Results

The following plane truss has been discretized in the FEM system Autodesk ROBOT, whose span has been assumed at the Gaussian input random variable. The following profiles have been proposed: (i) upper chord—SHS 140 × 140 × 8, (ii) diagonals—SHS 90 × 90 × 8, and (iii) lower chord—SHS 120 × 120 × 8. Its height has been adopted as h = 1.50 m and it was subjected to the uniformly distributed load, whose characteristic value was set as q_k = 15 kN/m, and the design load was taken for an illustration as q_d = 20 kN/m (load combinations do not affect further discussion).

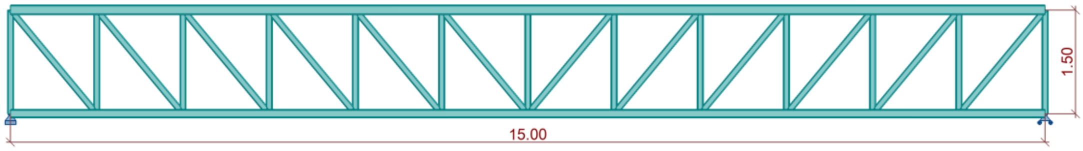

Figure 1: Static scheme of the given Pratt truss structure.

Preliminary results containing probabilistic relative probabilistic entropies due to the Bhattacharyya as well as due to the Kullback-Leibler models have been contrasted in Table 1 with the FORM reliability indices for the extreme deflection of the truss center line. One compares here the reliability indices resulting from (*i*) FEM analysis enriched with the Monte-Carlo scheme (MCS-NC), (*ii*) Monte-Carlo simulation prepared for the analytical formula (MCS-AC), (*iii*) stochastic perturbation-based FEM (PM-NC) as well as (*iv*) stochastic perturbation-based approach performed for the analytical deflection line (PM-AC). All probabilistic methods return the results very similar to each other for the given coefficient of variation of the truss span. It is also seen that certain numerical processing of relative entropies may result in relatively satisfactory coincidence of their values with the existing FORM indices. They all expectedly decrease together with an increasing input uncertainty level, however parametric sensitivity of these relative entropies may be different in specific engineering applications and deserves further numerical experiments. In case of a lack of coincidence of the specific relative entropies with the given FORM index variability interval, some additional numerical rescaling procedure should be proposed. Traditional analytical solutions in elasticity theory subjected to parameter randomization may serve as the reliable basis for such rescaling.

α [-]	Reliability method	MCS-NC	MCS-AC	PM-NC	PM-AC
0.025	B-entropy	38.11	38.43	38.11	38.43
	KL-entropy	4.681	5.00	4.68	5.00
	FORM index	53.45	53.81	53.45	53.81
0.050	B-entropy	19.05	19.21	19.05	19.21
	KL-entropy	4.66	4.98	4.66	4.98
	FORM index	13.38	13.47	13.38	13.47
0.075	B-entropy	12.69	12.79	12.69	12.79
	KL-entropy	4.64	4.95	4.64	4.95
	FORM index	5.99	6.04	5.99	6.04
0.100	B-entropy	9.50	9.58	9.50	9.58
	KL-entropy	4.59	4.91	4.60	4.91
	FORM index	3.44	3.47	3.44	3.47
0.125	B-entropy	7.59	7.61	7.59	7.65
	KL-entropy	4.55	4.86	4.55	4.85
	FORM index	2.29	2.32	2.29	2.32
0.150	B-entropy	6.31	6.32	6.31	6.37
	KL-entropy	4.50	4.79	4.49	4.79
	FORM index	1.70	1.72	1.70	1.72

Table 1: Comparison of the rescaled reliability indices by Bhattacharyya and Kullback-Leibler relative entropies by numerical (NC) and analytical methods (AC).

4 Conclusions and Contributions

[1] It has been demonstrated in this work that the given stochastic numerical methods may be generally applied to determine relative probabilistic entropies in many important engineering case studies. This is true in all these cases, where relative entropy may be analytically expressed using the parameters of uncertainty sources probability distribution functions. It is especially relatively easy for the Gaussian random parameters and this case study may be solved using the existing engineering FEM software with minor probabilistic modifications. Otherwise, one needs to apply computer algebra software enabling the symbolic derivation of such closed-form expressions, which may be difficult when the reliability limit contains two different probability distributions.
[2] A comparison of various relative entropies with the classical First Order Reliability Method fundamental for structural design generally confirms the applicability of these entropies in engineering reliability analysis. However, direct usage of their numerical values and comparison with various lower bounds suggested in many existing designing codes demands some scaling procedures varying upon relative entropy model. So that, some rescaling formulas describing modifications of the well-known formulas for relative entropies have been proposed in this work, which could be successively incorporated into designing civil engineering procedures (using Eurocode 0, for instance). It has been demonstrated that such a procedure is efficient while analyzing the linear and reversible structural static/dynamic processes.

[3] Common application of three different probabilistic numerical methods was possible thanks to the usage of the Response Function Method based upon global or local polynomial bases. They have been recovered thanks to the Least Squares Method via specific series of the FEM experiments with varying input parameters. This part needs only minor modifications of any of the existing FEM systems and may be replaced in the future with non-polynomial bases, which may appear to be more efficient in some specific case studies. The research presented in this work should be continued, especially towards structural stability of the thin-walled large scale structures, where a limit function demands some other numerical algorithms than in traditional quasi-static deterministic FEM analysis.

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