The Eleventh International Conference on
Engineering Computational Technology
23–25 August, 2022 | Montpellier, France

Proceedings of the Eleventh International Conference on
Engineering Computational Technology
Edited by B.H.V. Topping and P. Iványi
Civil-Comp Conferences, Volume 2, Paper 2.11
Civil-Comp Press, Edinburgh, United Kingdom, 2022, doi: 10.4203/ccc.2.2.11

Introducing Anisotropic Eddy-viscosity Coefficient with Single-equation Model

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Abstract

A one-equation turbulence model is formulated which retains an anisotropic eddy viscosity coefficient. Consequently, the current model is deemed to be potential for accounting near-wall turbulence and strong flow in-homogeneity, enhancing the predictive accuracy for complex separated and reattaching flows. Furthermore, the devised turbulence model retrieves the link to the k-ε model and is likely to be extendable toward a non-linear algebraic Reynold stress model. Intuitively, the current artifact may accommodate a variable eddy-viscosity coefficient for an LES (large eddy simulation) or a DES (Detached eddy simulation) method.

Keywords: one-equation model, near-wall turbulence, non-equilibrium flow, LES, DES.

1 Introduction

To replicate both equilibrium and non-equilibrium flow in one-equation models, considerable innovative research has been undertaken [1–7]. Empiricism and arguments of dimensional analysis are involved in the widely used one-equation Spalart and Allmaras (*SA*) model [1], avoiding the link to the traditional k-ε turbulence model. Internal and external flows are extensively utilized to calibrate and validate the SA model; providing reasonable predictions. However, connection to the k-ε model has been ameliorated with recently developed one-equation models by Rahman et al. [3–6]. They reproduce relatively improved predictions for separated and reattaching flows owing to their capability in accounting for non-equilibrium effects via variable model coefficients. In fact, a single-equation turbulence model establishes a good compromise between algebraic and two-equation models because of inheriting transport effects.

Customarily, one/two-equation models encounter non-equilibrium effects when being embedded with an anisotropic eddy-viscosity coefficient C_μ, parameterized with a production to dissipation ratio P_k/ε accompanied by invariants of mean strain-rate and vorticity tensors. The resulting C_μ suppresses non-physical energy components at moderate/severe strain rates on the perspective of realizability constraint, representing a minimal requirement for the turbulence model. Therefore, the current eddy-viscosity formulation with an appropriate strain-dependent C_μ reinforces turbulence anisotropy in a single-equation model. In addition, *k* and ε are explored in the present model, reviving presumably the competency in speculating complex separated and reattaching flows.

2 Formulation of present turbulence model

A transport equation for $R = C_\mu k^2/\varepsilon$ (pseudo-eddy viscosity) can be obtained using the two-equation *k*-ε turbulence model. The following relation is used to construct an *R-transport* equation:

$$\frac{DR}{Dt} = \frac{D\left(C_\mu k^2/\varepsilon\right)}{Dt} = C_\mu\left(\frac{2k}{\varepsilon}\frac{Dk}{Dt} - \frac{k^2}{\varepsilon^2}\frac{D\varepsilon}{Dt}\right) = \frac{2R}{k}\frac{Dk}{Dt} - \frac{R}{\varepsilon}\frac{D\varepsilon}{Dt} \tag{1}$$

where the substantial derivative is indicated by *D/Dt*. Equations of *k* and ε at a high Reynolds number can be provided with:

$$\frac{Dk}{Dt} = P_k - \varepsilon + \nabla.\left(\frac{R}{\sigma_k}\nabla k\right) \tag{2}$$

$$\frac{D\varepsilon}{Dt} = C_{\varepsilon 1}\frac{\varepsilon}{k}P_k - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \nabla.\left(\frac{R}{\sigma_\varepsilon}\nabla\varepsilon\right) \tag{3}$$

where P_k implies the production term; relevant model constants are σ_k, σ_ε, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ Combining Equations (1)-(3) and carrying out some algebra with $\sigma_k = \sigma_\varepsilon = \sigma_R$, result in an *R-transport* equation:

$$\begin{aligned}\frac{DR}{Dt} = &\left(2 - C_{\varepsilon 1}\right)\frac{R}{k}P_k - \left(2 - C_{\varepsilon 2}\right)k + \frac{\partial}{\partial x_j}\left[\left(\frac{R}{\sigma_R}\right)\frac{\partial R}{\partial x_j}\right] - \frac{2R^2}{k^2\sigma_R}\frac{\partial k}{\partial x_j}\frac{\partial k}{\partial x_j} \\ &+ \frac{4R}{k\sigma_R}\frac{\partial k}{\partial x_j}\frac{\partial R}{\partial x_j} - \frac{2}{\sigma_R}\frac{\partial R}{\partial x_j}\frac{\partial R}{\partial x_j}\end{aligned} \tag{4}$$

Apparently, the diffusion/destruction term $\left(\frac{4R}{k\sigma_R}\frac{\partial k}{\partial x_j}\frac{\partial R}{\partial x_j} - \frac{2R^2}{k^2\sigma_R}\frac{\partial k}{\partial x_j}\frac{\partial k}{\partial x_j}\right)$ appearing in the above-mentioned relation may be excluded in order to avoid the numerical stiffness. Therefore, Equation (4) can be regularized using the Bradshaw relation $\left|-uv\right| = \sqrt{C_\mu} k = R\left|\frac{du}{dy}\right|$ [8] with the k-ε source and sink terms:

$$\frac{D\rho R}{Dt} = C_1 \rho R \tilde{S} - C_\mu^* \rho k/4 + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\sigma_R}\right)\frac{\partial R}{\partial x_j}\right] - C_2 \rho \frac{\partial R}{\partial x_j}\frac{\partial R}{\partial x_j} \tag{5}$$

where $C_1 = \sqrt{C_\mu\left(C_{\varepsilon 2} - C_{\varepsilon 1}\right)}$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.9 + C_\mu/4$, $C_\mu = 0.09$, $C_2 = 2/\sigma_R$, and $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.9 + C_\mu/4$, $C_\mu = 0.09$, $C_2 = 2/\sigma_R$ and $\sigma_R = 1.3$ The Park-Park limiter [9] is applied to determine the eddy-viscosity μ_T :

$$\mu_T = \rho \min\left[\sqrt{\frac{2}{3}}\frac{kT_t}{\zeta}; \min\left(f_{v1}R; C_\mu^* kT_t\right)\right] \tag{6}$$

where the hybrid time scale T_t can be given by [5]:

$$T_t = \sqrt{\frac{k^2}{\varepsilon^2} + C_T^2 \frac{\nu}{\varepsilon}} = \frac{k}{\varepsilon}\sqrt{1 + \frac{C_T^2}{\mathrm{Re}_T}}, \ \mathrm{Re}_T = \frac{k^2}{\nu\varepsilon} \tag{7}$$

where Re_T denotes the turbulent Reynolds number, $C_T = \sqrt{2}$ is an empirical constant and ν ($= \mu/\rho$; ρ is the density and μ is the molecular viscosity) signifies the kinematic viscosity.

The mean strain-rate S_{ij} and vorticity W_{ij} tensors, required afterward can be defined By

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \ W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{8}$$

The invariants of vorticity and mean strain-rate tensors can be represented by $W = \sqrt{2W_{ij}W_{ij}}$ and $S = \sqrt{2S_{ij}S_{ij}}$, respectively. The eddy-viscosity coefficient C_μ^* in Equation (6) has been formed with mean strain-rate and vorticity invariants; C_μ^* as suggested in Reference [5] has been adopted:

$$C_\mu^* = \frac{\alpha_1}{1 - \frac{3}{2}\eta^2 + 2\xi^2} \tag{9}$$

where non-dimensional mean shear strain-rate and mean rotation-rate parameters are defined by $\eta = \alpha_2 T_t S$ and $\xi = \alpha_3 T_t W$, respectively. Coefficients α_1-α_3 in Equation (9) are given by:

$$\alpha_1 = g\left(\frac{1}{4} + \frac{2}{3}\sqrt{\Pi_b}\right), \qquad \alpha_2 = \frac{3}{8\sqrt{2}} g$$
$$\alpha_3 = \frac{3}{\sqrt{2}}\alpha_2, \qquad g = \left(1 + 2\frac{P_k}{\varepsilon}\right)^{-1} \tag{10}$$

with $\Pi_b = b_{ij}b_{ij}$; the Reynolds stress anisotropy b_{ij} is characterized by

$$b_{ij} = \frac{\overline{u_i u_j}}{2k} - \frac{1}{3}\delta_{ij} \tag{11}$$

Rahman et al. [3–6] have devised three explicit solutions to P_k/ε. The current formulation utilizes the simplest one, indicating a universal value of $\sqrt{\Pi_b} \approx 0.31$ in homogeneous turbulence. In addition, P_k/ε receives an approximated consistency condition with $P_k/\varepsilon = \zeta/3.2$ in homogeneous turbulence, where $\zeta = T_t S \, max\,(1, \Re)$, and $\Re = |W/S|$ indicates a non-dimensional variable [5].

The associated quantity f_{v1} in Equation (6) represents an eddy-damping function, defined by

$$f_{v1} = \left[1 - \exp\left(-\frac{y}{L}\right)\right]^2, \; L^2 = \zeta\left(7.1 + C_\mu^* \, \mathrm{Re}_T\right)\sqrt{\frac{v^3}{\varepsilon}} \tag{12}$$

where y implies a wall-distance parameter usually normal to the wall and $\left(v^3/\varepsilon\right)^{1/4}$ signifies the Kolmogorov length scale.

The tensor $\tilde{S}$ linked to the production term in Equation (5) represents the scalar measure of deformation; unlike the *SA* turbulence model [1], this term is redefined in order to take the effect of vorticity into account:

$$\tilde{S} = f_k\left(S - \frac{|\eta_1| - \eta_1}{2}\right), \; f_k = 1 - \frac{f_{v1}}{2}\sqrt{\max\left(1 - \Re, 0\right)} \tag{13}$$

where $\eta_1 = S - W$. Equation (13) has a close resemblance to an enhancement approach for the turbulence model sensitivity toward the effect of streamline curvature, provoking an extra rate of strain over and above the main strain-rate in the flow field. The kinetic energy of turbulence k can be approximated with the aid of Bradshaw's relation as [11]:

$$k = f_{v1}^{0.3}\frac{RS_k}{\sqrt{C_\mu}}, \; S_k = \sqrt{\tilde{S}^2 + S_\alpha^2} \tag{14}$$

The non-vanishing strain-rate correction term S_α in the free-stream region can be designed using the log-law behaviour of pseudo-eddy viscosity $R = u_\tau k y$ and $du/dy = u_\tau/ky$ as:

$$S_\alpha = f_{v1} \max\left[\left(\frac{3}{2\kappa}\frac{\partial\sqrt{R}}{\partial x_i}\right)^2; \frac{1}{C_\mu}s^{-1}\right] \tag{15}$$

where $1/C_\mu \, s^{-1}$ has been estimated from a nearly homogeneous shear flow [10] and the von-Karman constant $\kappa = 0.41$. The hybrid time scale T_t requires an evaluation of the total dissipation-rate ε since it plays an important role in constructing a compatible T_t; ε is determined as follows [11]:

$$\varepsilon = \sqrt{\varepsilon_\omega^2 + \tilde{\varepsilon}^2}\;, \; \tilde{\varepsilon} = RS_k^2 f_{v1}^{1.3} \tag{16}$$

where unlike ε, $\tilde{\varepsilon}$ disappears at the solid wall due to the product $\left(R\times f_{v1}^{1.3}\right)$. However, ε_w indicates the wall-dissipation rate, balanced by the viscous-diffusion rate at the wall vicinity; ε_w is conventionally modelled as:

$$\varepsilon_w = 2A_\varepsilon \nu \left(\frac{\partial u}{\partial y}\right)_w^2 \approx 2A_\varepsilon \nu S_k^2 \tag{17}$$

where $A_\varepsilon = C_\mu = 0.09$ from *DNS* data. Apparently, the total dissipation-rate ε is likely to be benefited by the wall dissipation-rate ε_w within the wall-layer.

3 Results

Fully-developed turbulent channel flows at $Re_\tau = 180$, *395* and *640* are simulated to substantiate the model accuracy in replicating near-wall turbulence. Computations are carried out in a half-width h of the channel using a *1-D* (one-dimensional) *RANS* solver. A non-uniform *1×64* grid resolution for $Re_\tau = (180; 395)$ and *1×128* grid resolution for $Re_\tau = 640$ are assumed to be adequate to accurately describe characteristics of the flow. To assure the viscous sublayer resolution, the first near-wall grid spacing is set to $y^+ \approx 0.3$. A cell-centered finite-volume approach is applied to solve the flow equations. Results are converted to the form of $u^+ = u/u_\tau$, $k^+ = k/u_\tau^2$, $\overline{uv}^+ = \overline{uv}/u_\tau^2$, $\varepsilon^+ = \nu\varepsilon/u_\tau^4$, where u_τ is the wall-friction velocity; comparisons are made by plotting these quantities versus $y^+ = yu_\tau/\nu$. Turbulence quantities are extracted from *DNS* data [12, 13]. Predictions of the present model are compared with those of the widely-used *SA* turbulence model [1].

The stream-wise mean *x*-momentum equation for a *1-D* incompressible flow can be represented by

$$\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\left(\nu + \nu_T\right)\frac{\partial u}{\partial y}\right] = 0 \tag{18}$$

where $\nu_T = \mu_T/\rho$ is the kinematic eddy-viscosity; lower and upper wall locations of the channel are encompassed by $y = (-h; h)$. The axial pressure-gradient $\partial p/\partial x$ remains constant and the continuity constraint $\partial u_i/\partial x_i = 0$ is naturally satisfied, as the mean flow field has a *1-D* feature. However, $\partial p/\partial x$ must be computed as a part of the solution method since the pressure gradient is not known a priori. The pressure-velocity correction *(PVC)* method [14, 15] is an appropriate choice to solve the problem. The *PVC* scheme keeps updating the axial pressure gradient and velocity as long as the fictitious mass source is minimized.

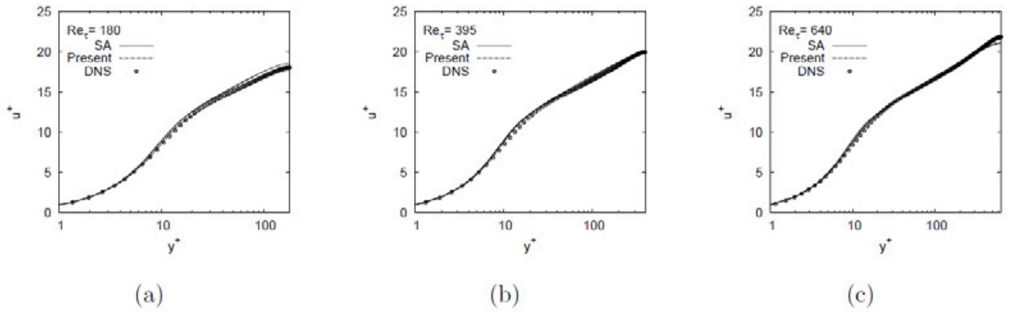

Figure 1: Velocity profiles for fully-developed turbulent channel flow.

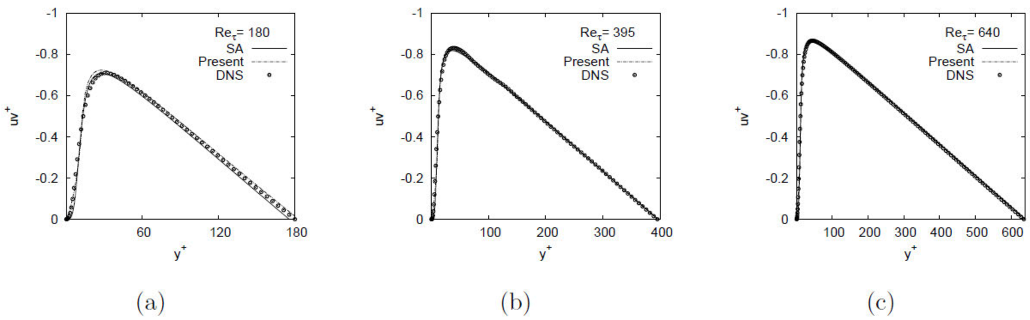

Figure 2: Shear stress profiles for fully-developed turbulent channel flow.

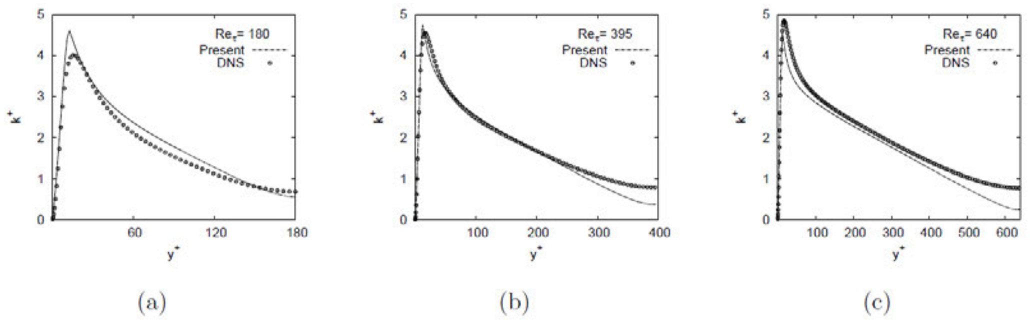

Figure 3: Kinetic energy profiles for fully-developed turbulent channel flow.

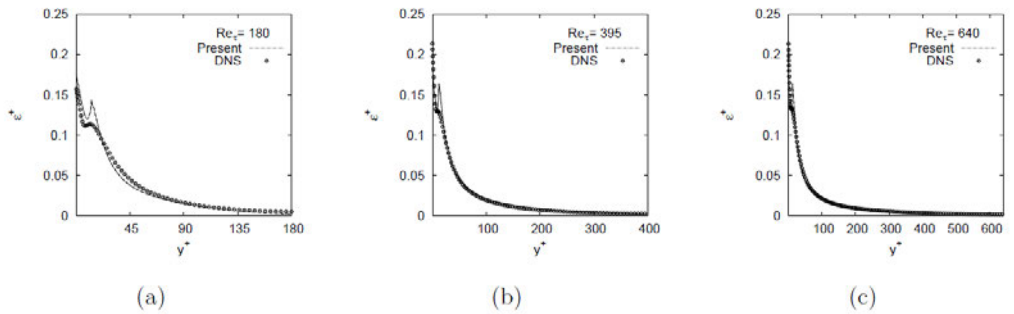

Figure 4: Dissipation-rate profiles for fully-developed turbulent channel flow.

Predicted profiles of the velocity and turbulent shear-stress from independent turbulence models are illustrated in Figures 1 and 2, respectively. It seems likely that indistinguishable predictive performances pertaining to both models are obtained. As can be seen, two turbulence models make pretty good correspondence with *DNS* data in both regions, comprising the linear boundary layer and wake defect layer. Present model performances are further assessed with turbulent kinetic energy k^+ and dissipation-rate ε^+ profiles as shown in Figures 3 and 4, respectively. Note-worthily, reasonable agreement of the current model with *DNS* data is visible without having transport and diffusion effects of the turbulent kinetic energy and dissipation-rate. It appears that the near-wall k^+-profile is qualitatively well reproduced and the maximum magnitude of ε^+ is captured in the wall-vicinity, as dictated by *DNS* and experimental data.

4 Conclusions

A compatible eddy-viscosity coefficient is introduced with the current model, the potential importance of which is not obvious since only a fully-developed turbulent channel flow case (e.g., simple shear flow case) is computed for validation. However, it is believed that the modification is profoundly convenient to account for strong flow in-homogeneity and near-wall turbulence and therefore, it can enhance the model competency in speculating complex separated and reattaching flows to a greater extent. Articulately, the link to the $k-\varepsilon$ model with the present model is retrieved and likely to be extendable toward a non-linear algebraic Reynolds stress model. Intuitively, the present formulation may accommodate a variable eddy-viscosity coefficient for an *LES* or a *DES* method.

Acknowledgements

This work is supported by Hangzhou Dianzi University research supporting fund (Grant No. GK218803299037) of Zhejiang Province, P.R. China

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