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A Computationally Efficient Hybrid Magnetic Field Correction for the Magnetohydrodynamic Equations

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Abstract

During the simulations of the magnetohydrodynamic equations, numerical errors might cause the formation of non-physical divergence components in the magnetic field. This divergence compromises the stability and accuracy of the simulations. In order to overcome this problem, several methodologies, called divergence cleaning methods, are proposed. Besides many comparative works between these methods, the construction of the best approach is still an open problem. A popular divergence cleaning strategy is the parabolic-hyperbolic approach due to its easy implementation and low computational cost in CPU time, however this approach just transports and diffuses the divergence components instead of eliminating them globally. On the other hand, the elliptic approach, also known as the projection method, uses a Poisson equation to eliminate the divergence effectively at a huge computational cost. This work proposes a successful combination of these approaches in order to create a new divergence cleaning methodology that incorporates the advantages provided by both methods, a small CPU time and a good accuracy.

Keywords: adaptive mesh refinement, magnetohydrodynamics, high performance computing, divergence cleaning.

1 Introduction

In recent years, the study of phenomena associated with the interplanetary environment has been increasing due to the significant consequences of those effects on many modern communication infrastructure systems [1]. These studies depend on both the analysis of the data collected from several instruments and numerical simulations of models such as the magnetohydrodynamic (MHD) one, which describes the plasma dynamo as a magnetised fluid.

In particular, the numerical errors obtained during the MHD simulations cause the formation of non-physical divergence components in the magnetic field that may compromise the stability and the accuracy of the simulation. Therefore, a methodology capable of correcting the solution after every iteration of the time evolution process is required to maintain the validity of Gauss's law for magnetism in the solution [2].

Several strategies, named divergence cleaning methods, are proposed in the literature in order to overcome this problem [3,4]. In the finite volume context, these strategies can be classified into two types: the ones that are characterized by lower computational cost and eliminate the divergence components to a sufficient extent as to not compromise the solution of the physical phenomena, e.g., the parabolic-hyperbolic correction introduced in [3], and the ones that remove these divergence components entirely requiring a high computational cost and induce slight diffusion in the solution, e.g., the projection method introduced in [2].

The efficiency of different divergence cleaning methods has been compared in several studies [5,6]. However, construction of an optimal divergence cleaning approach is still an open problem considering the high performance computing scope. This work presents a new approach that combines the parabolic-hyperbolic correction with the projection method using the Generalised Lagrange Multiplier (GLM) context introduced in [3] and [7]. This combination aims to extract the advantages of the categories of both approaches, obtaining a method that effectively eliminates the divergence without drastically increasing the computational cost.

2 Methods

The equations for the ideal MHD model are given by the following system of equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \left(p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) \mathbb{I} - \mathbf{B}\mathbf{B} \right] = 0$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left[\left(\mathcal{E} + p + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot [\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}] = 0, \tag{1}$$

with

$$\mathcal{E} = \frac{p}{\gamma - 1} + \rho \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{\mathbf{B} \cdot \mathbf{B}}{2}, \tag{2}$$

The parabolic-hyperbolic approach introduced by [3] uses a formulation employing Lagrange multipliers to introduce a scalar field in the induction equation in order to transport and diffuse the divergence components. Therefore, the induction equation becomes

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u} + \psi \mathbb{I}) = \mathbf{0}, \tag{3}$$

where the new scalar field is ruled by the equation

$$\frac{\partial \psi}{\partial t} + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2} \psi. \tag{4}$$

Therefore the hyperbolic and parabolic contributions are expressed by indexes h and p, respectively.

The projection method introduced in [2] can also be expressed in the GLM formulation, obtaining the elliptic correction [4]. This method consists in solving a Poisson equation to associate the magnetic field with divergence components to a scalar field so that

$$\nabla^2 \psi = \frac{1}{\Delta t} \nabla \cdot \hat{\mathbf{B}}. \tag{5}$$

After that, the divergence free magnetic field is corrected during the time integration using the relation

$$\mathrm{B}^{n+1} = \hat{\mathrm{B}} - \Delta t \nabla \psi. \tag{6}$$

This work evaluates the combination of the explicit parabolic-hyperbolic and the implicit elliptic approaches. The proposed method, denominated GLM triple correction, consists in performing the parabolic-hyperbolic approach, that produces a small divergence over the magnetic field, until a cleaning criteria is fulfilled. Then, the elliptic method is applied in order to restore the divergence constraint. The cleaning criteria chosen is the application of the elliptic operator after a predefined number of time iterations. These approaches are implemented in the MHD module of the AMROC framework [8,9,10,11].

3 Results

The discussed divergence cleaning approaches are tested using several traditional MHD benchmarks: Magnetic field loop advection (ADV); Rotor (ROT); Orszag-Tang vortex (OTV); Spherical Blast Wave (BWV), with usual initial and boundary conditions and final time. These MHD simulations were performed using the HLLD numerical flux with the MC limiter, Courant number of 0.4 and the parameter of ratio between the parabolic-hyperbolic correction is 0.4. The MHD time evolution method is a second-order accurate Runge-Kutta scheme. The triple correction executes the elliptic operation every 60 iterations.

Figure 1 presents the ratio between the divergence and the magnitude of the magnetic field obtained by the discussed methods for each problem. The parabolic-hyperbolic approach obtains machine zero precision in some regions of the ROT and BWV problems due to the divergence errors not being transported to those regions yet. In general, the elliptic approach presents the lowest divergence errors, specially in the ADV and BWV problems. However, it exhibits the highest divergence peak in the ROT problem, besides presenting lower divergence globally. The proposed triple correction yields lower divergence errors than the parabolic-hyperbolic correction, especially in the regions containing structures, such as the centre of the rotor and in the inner parts of the blast wave.

The absolute values for the divergence obtained in each case are presented in Table 1 alongside with the correspondent CPU time. The results confirm the reduction in the divergence provided by the triple correction in relation to the parabolic-hyperbolic correction. Moreover, just a slight CPU time increase is observed. In terms of precision, the triple correction is comparable with the elliptic operator. However, it reduces the CPU time significatively.

As the ADV problem has an exact solution, the error obtained for the components Bx using each approach are presented in Table 2. The triple correction presents the best result in both norms. On the other hand, the parabolic-hyperbolic correction produces a significantly higher error using the L1 norm. In Figure 2, we plot the magnitude of the magnetic field, and we can observe that the triple correction gives the best quality, without the spurious oscillations and background noise of the parabolic-hyperbolic approach and is less diffusive than the elliptic approach.

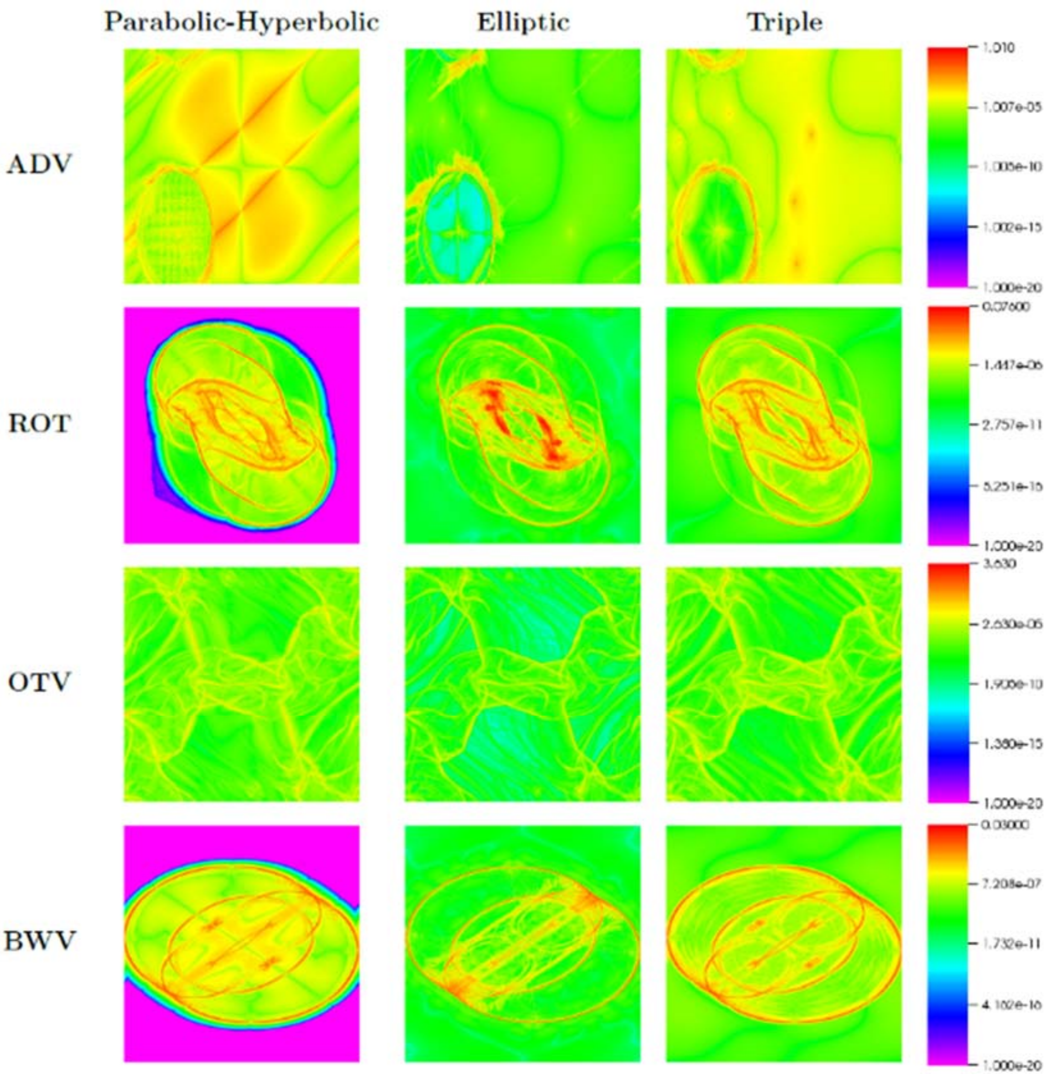

Figure 1: Parameter $dx\frac{\nabla\cdot B}{|B|}$, obtained for the 2D problems using the discussed divergence cleaning approaches in a mesh with 2048^2 cells.

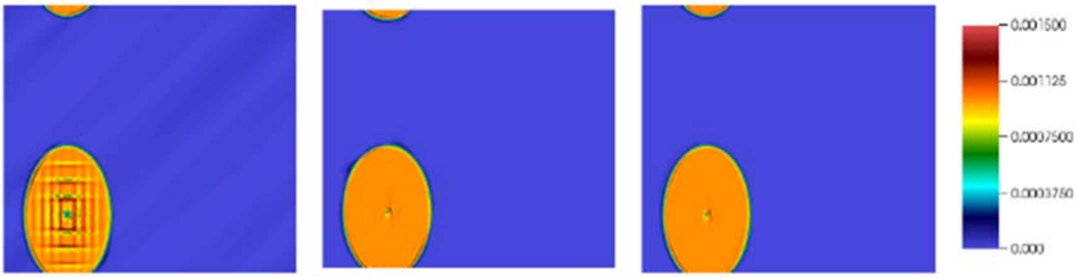

Figure 2: ADV Problem: $||B||$ obtained using a mesh with 2048^2 cells using the discussed divergence cleaning approaches.

	Method	$\|\|\nabla \cdot B\|_1\|$	$\|\|\nabla \cdot B\|_\infty\|$	t_{CPU}
ADV	Par-Hyp	0.20×10^{-6}	3.99×10^{-3}	9.47
	Elliptic	2.34×10^{-6}	4.18×10^{-3}	39.47
	Triple	5.06×10^{-6}	2.40×10^{-3}	12.05
ROT	Par-Hyp	1.95×10^{-1}	1.42×10^{2}	0.67
	Elliptic	3.71×10^{-1}	2.40×10^{2}	2.70
	Triple	1.85×10^{-1}	1.35×10^{2}	0.67
2D-OTV	Par-Hyp	1.55×10^{0}	2.24×10^{1}	1.51
	Elliptic	0.56×10^{0}	0.91×10^{1}	5.55
	Triple	1.37×10^{0}	1.86×10^{1}	2.30
BWV	Par-Hyp	2.00×10^{-1}	6.79×10^{1}	0.65
	Elliptic	1.05×10^{-1}	3.84×10^{1}	2.81
	Triple	1.85×10^{-1}	6.23×10^{1}	0.74

Table 1: Divergence and CPU time, in hours, obtained by the discussed divergence cleaning methods.

	Norm	Method		
		Par-Hyp	Elliptic	Triple
B_x	$L_1(x10^{-6})$	33.57	7.46	6.91
	$L_\infty(x10^{-4})$	11.44	10.25	9.94

Table 2: ADV Problem: Errors for B_x component in the L_1 and L_∞ norms obtained by the studied divergence cleaning approaches in the mesh with 2048^2 cells using 16 processors.

4 Conclusions and Contributions

As a new contribution to the accuracy and quality of the solutions provided by the AMROC framework MHD solver, this work presents a successful new divergence cleaning approach that combines the parabolic-hyperbolic correction introduced in [3] and the elliptic correction introduced [4], creating a triple parabolic-hyperbolic-elliptic correction that applies a multigrid strategy to overcome the performance limitations of the elliptic operator. This correction leads to a reduction in the global divergence of the magnetic field without compromising the solver performance.

Further studies into optimisations of the cleaning criteria for the triple correction are ongoing for both two and three dimensional benchmarks.

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