Proceedings of the Sixth International Conference on
Railway Technology: Research, Development and Maintenance
Edited by: J. Pombo
Civil-Comp Conferences, Volume 7, Paper 5.13
Civil-Comp Press, Edinburgh, United Kingdom, 2024
ISSN: 2753-3239, doi: 10.4203/ccc.7.5.13

Dynamic Analysis of a Ballast-Less Railway Track Through a Periodically Supported Timoshenko Beam and Dynamical Forces Consideration

D. Gonzalez Velasco[1], D. Duhamel[1], T. Hoang[1], G. Foret[1], H. Lenglin[2] and B. Findinier[2]

[1] Laboratoire Navier, Ecole Nationale des Ponts et Chaussees, Champs-sur-Marne, France
[2] GETLINK, Coquelles, France

Abstract

This article focuses on the extension of a ballast-less railway track analytical model to compute its dynamic responses due to the application of forces on the structure. Similarly to various existing models, the railway track is represented as a periodically supported Timoshenko beam on individual and independent supports. The supports consist of a two-stage damping spring system with a mass in between. The structure is subjected to mobile forces, representing the passage of trains, which move along the railway track at a constant speed. In existing models, the structure is subjected to static forces, representing trains as a moving mass. Since dynamic forces, resulting from wheel or track defects, can turn into the most important source of loads that can stress the structure, a new formulation of the force is introduced with the aim of taking into account the influence of dynamic forces on the behavior of the railway track. This new approach aims to improve the understanding of the railway track's response to a load that seeks to be more complete and faithful to real-world phenomena.

Keywords: ballast-less railway track, analytical model, Timoshenko's beam theory, static and dynamic loads, moving forces, dynamical reponse.

1 Introduction

Railway track modeling is of great interest in addressing the need to assess the structure's response when subjected to various loads, primarily caused by train transit. Within the existing literature, different approaches to railway track modeling can be found, stemming from either numerical or analytical modeling perspectives. Various possibilities are explored, such as approximating a railway track to a periodically supported beam for the analytical case and loading it with point or moving loads.

Analytical models are known for their efficiency in computing calculations within a short time span. In the railway sector, they are particularly valuable for structural design since they can provide the structure's response in terms of reaction forces and displacement within seconds and can be adapted to study multiple railway track configurations. This approach has been adopted by multiple authors [1–6], who represent the railway track as a periodically supported beam using Euler's and Timoshenko's beam theories. Loading is introduced through a static mobile force, aiming to represent the train's weight moving at a constant speed.

Among the types of loads to which a railroad track may be subjected, static loads have been accurately addressed in the works from Hoang [4, 5] and Claudet [1]. On the other hand, dynamic loads, resulting from wheel or track defects, are often viewed as detrimental to the structure's health and are not considered yet in existing analytical models, to our knowledge.

Through this article, the aim is to extend the scope of analytical models of periodically supported beams by introducing not only static but also dynamic loads to obtain the response of a ballastless railway track. Initially, we will address the proposed ways to model the different components interacting in the problem, and then present the complete development of the analytical model based on the Timoshenko's beam model. Finally, examples of model applications will be presented where a railway structure will be subjected to static and/or dynamic loads to evaluate its behavior.

2 Railway track analytical model development

2.1 Problem Description

The objective here is to model the behavior of a ballastless railway track. Since the problem is treated from an analytical point of view, it will be considered as a one-dimensional (1D) problem. For the modeling of the railway track we consider a general problem, which has the advantage of being adaptable according to the configuration one wishes to study. That is to say, as shown in Figure 1, the railway track is divided into patterns, all of which are identical and periodic. These patterns are characterized by having a finite number of point supports m, which may or may not be spaced the same distance apart from each other and have different mechanical char-

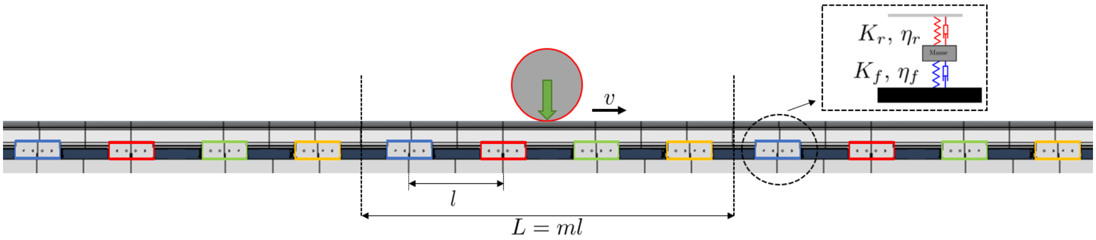

Figure 1: Railway track configuration subjected to moving loads. This specific system contains identical and equally spaced supports within the section L composed by $m = 4$ different supports having distinct mechanical properties represented by the various colours.

acteristics among them. The modeling choice starts by dividing the structure into it's different components: the supports, the rail, and the loads.

The supports will be modeled as independent and ponctual systems, as shown in figure 1, capable of exhibiting linear (or nonlinear) behavior. This independent modeling fact, facilitates the representation of railway tracks containing homogeneous or non-homogeneous sections. $k_{s,p}(\omega)$ being an equivalent stiffness of the system, it can be expressed in terms of $k_{r,p}(\omega)$ and $k_{f,p}(\omega)$, which are parameters taking into account the rail pad stiffness and the under-sleeper stiffness respectively, as shown in figure 1. This formulation is written as follows,

$$\frac{1}{k_{s,p}(\omega)} = \frac{1}{k_{r,p}(\omega)} + \frac{1}{k_{f,p}(\omega)} \tag{1}$$

Regarding the forces acting on the system, they will be categorized into two components. Firstly, there will be static loads representing the weight of the train, supplemented by dynamic loads represented as a function of space. These loads will move along the track at a constant speed v, with each load separated from the next by a distance D_j, which refers to the spacing between each pair of wheels composing a train.

Thus, the total force applied to the system under examination is expressed in Equation 2, where the first term accounts for the reaction forces generated by the supports, and the second term represents the forces due to the passage of trains.

$$F(x,t) = \sum_{n\in Z}\sum_{p=0}^{m-1} R_p(t-\frac{nL}{v})\delta(x-pl-nL) - \sum_{j=1}^{K}(Q_{j,stat}+Q_{j,dyn}(x))\delta(x+D_j-vt) \tag{2}$$

From the last expression, and as shown in figure 1, R_p corresponds to the reaction support forces, L to a section length, l to the distance between consecutive supports and Q_j to the force contribution attributed to the train transit. Q_j is then divided into its static and dynamic component. $Q_{j,stat}$ corresponds to a constant force, so it comes to a term remaining the same over space and time, whereas $Q_{j,dyn}$ is a term consisting on an amplitude (Q_d) of a force times a function in space describing the force's application behavior. Expressing the dynamic term of the force yields,

$$Q_{j,dyn}(x) = \sum_{n \in \mathrm{Z}} Q_{d,j} \; f_j(x - nL) \tag{3}$$

f_j is a function defined in $[\frac{-L}{2}, \frac{L}{2}]$, describing the j^{th} force distribution and is null outside this domain.

On the other hand, the rail will be modeled using the Timoshenko beam theory, which is suitable for analyzing a wide range of frequencies and provides results closer to reality when the train's speed is significant and the spacing between supports is small [5]. The dynamic equation governing the behavior of the beam is as follows,

$$\begin{cases} \rho S \frac{\partial^2 w_r(x,t)}{\partial t^2} = \kappa S G \left(\frac{\partial^2 w_r(x,t)}{\partial x^2} - \frac{\partial \phi_r(x,t)}{\partial x} \right) + F(x,t) \\ \rho I \frac{\partial^2 \phi_r(x,t)}{\partial t^2} = EI \frac{\partial^2 \phi_r(x,t)}{\partial x^2} + \kappa S G \left(\frac{\partial w_r(x,t)}{\partial x} - \phi_r \right) \end{cases} \tag{4}$$

Where E, I, w_r, ϕ_r, ρ, S, and κ represent, respectively, the Young's modulus, the moment of inertia, the vertical displacement of the beam, the rotation of the beam's section, the mass density, the section area, and the shear factor of the beam's section.

2.2 Dynamic state equations development

The method described here reduces the number of unknowns by employing the assumption of "periodic conditions". This means that due to the system's periodicity, the response of a support m located at $x = 0$ will be identical to that of a support located at a distance L, as they possess the same equivalent stiffness ($K_q = K_{q+nm}, \; \forall n \in \mathrm{Z}$) and experience similar loading. The response of the support situated at L will consequently exhibit a time delay of $\frac{L}{v}$ relative to that of the support at $x = 0$. This periodic condition is extended to the force application term, which shall be spacially and temporally periodic to garanty a proper stablishment of the problem. Accounting for the periodicity of the system, the relationship between reaction forces among supports of different placements and the forces applicated to the system is respectively expressed as follows,

$$\begin{cases} R_{nm+p}(t) = R_p(t - \frac{nL}{v}), \; \forall n \in \mathrm{Z} \\ F(x,t) = F(x - nL, t - \frac{nL}{v}), \; \forall n \in \mathrm{Z} \end{cases} \tag{5}$$

$R_p(t)$, situated in the domain $0 \leq p < m$ and at the coordinate $x = pl$, denotes the reaction force of support p.

In addressing the problem, Hoang and Claudet employed an approach, that will be adopted here as well, relying on the Fourier series decomposition of periodic functions that depict the phenomenon.

In order to solve the system of equations proposed in 4, Hoang et al. [4] propose applying the Fourier transform with respect to t, and then to use the Fourier series expansion with respect to x. Timoshenko's beam dynamic equation Fourier transform with respect to t results in,

$$\begin{cases} \kappa SG\frac{\partial\hat{\phi}_r(x,\omega)}{\partial x} = \kappa SG\frac{\partial^2\hat{w}_r(x,\omega)}{\partial x^2} + \rho S\omega^2\hat{w}_r(x,\omega) + \hat{F}(x,\omega) \\ -\kappa SG\frac{\partial\hat{w}_r(x,\omega)}{\partial x} = EI\frac{\partial^2\hat{\phi}_r(x,\omega)}{\partial x^2} + (\rho I\omega^2 - \kappa SG)\hat{\phi}_r(x,\omega) \end{cases} \tag{6}$$

Furthermore, by following the same procedure for equation 2, and introducing $x_p = pl$, it becomes,

$$\frac{\hat{F}(x,\omega)}{\mathrm{e}^{-\mathrm{i}\omega\frac{x}{v}}} = \Bigg(\sum_{n\in \mathrm{Z}}\sum_{p=0}^{m-1}\hat{R}_p(\omega)\mathrm{e}^{\mathrm{i}\omega\frac{x_p}{v}}\delta\left(x - nL - x_p\right) - \sum_{j=1}^{K}\frac{\left(Q_{j,stat} + \sum_{n\in \mathrm{Z}} Q_{d,j}\, f_j(x-nL)\right)}{v}\mathrm{e}^{-\mathrm{i}\omega\frac{D_j}{v}}\Bigg) \tag{7}$$

Thus, we can notice that $\hat{F}(x,\omega)\mathrm{e}^{\mathrm{i}\omega\frac{x}{v}}$ is L-periodic. So by setting,

$$\hat{w}_r(x,\omega) = \Psi(x,\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}x} \quad \text{and} \quad \hat{\phi}_r(x,\omega) = \Phi(x,\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}x} \tag{8}$$

It is possible to rewrite equation 6. Therefore, it becomes,

$$\begin{aligned} \kappa SG\left(\frac{\partial\Phi}{\partial x} - \frac{\mathrm{i}\omega}{v}\Phi\right) &= \kappa SG\left(\frac{\partial^2\Psi}{\partial x^2} - 2\frac{\mathrm{i}\omega}{v}\frac{\partial\Psi}{\partial x} - \frac{\omega^2}{v^2}\Psi\right) + \rho S\omega^2\Psi + \mathrm{e}^{\mathrm{i}\frac{\omega}{v}x}\hat{F}(x,\omega) \\ \kappa SG\left(\frac{\mathrm{i}\omega}{v}\Psi - \frac{\partial\Psi}{\partial x}\right) &= EI\left(\frac{\partial^2\Phi}{\partial x^2} - 2\frac{\mathrm{i}\omega}{v}\frac{\partial\Phi}{\partial x} - \frac{\omega^2}{v^2}\Phi\right) - \left(\kappa SG - \rho I\omega^2\right)\Phi \end{aligned} \tag{9}$$

At this point, by performing a Fourier series expansion of $\Phi(x,\omega)$ and $\Psi(x,\omega)$ it is possible to find a solution to the system expressed in equation 9, by using Floquet's theorem. The Fourier series expansion of $\Phi(x,\omega)$ and $\Psi(x,\omega)$ takes the following form,

$$\Psi(x,\omega) = \sum_{n=-\infty}^{\infty} p_n(\omega)\mathrm{e}^{\mathrm{i}2\pi n\frac{x}{L}} \quad \text{and} \quad \Phi(x,\omega) = \sum_{n=-\infty}^{\infty} q_n(\omega)\mathrm{e}^{\mathrm{i}2\pi n\frac{x}{L}} \tag{10}$$

Where p_n and q_n are the Fourier coefficients for $\Psi(x,\omega)$ and $\Phi(x,\omega)$ respectively.

Moreover, equation 7 being L-periodic, its Fourier coefficients are computed by setting,

$$\frac{1}{L}\int_{-L/2}^{L/2}\hat{F}(x,\omega)\mathrm{e}^{\mathrm{i}\omega\frac{x}{v}}\mathrm{e}^{-2\mathrm{i}\pi n\frac{x}{L}}\,\mathrm{d}x = \frac{1}{L}\sum_{p=0}^{m-1}\hat{R}_p(\omega)\mathrm{e}^{\mathrm{i}\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)x_p} - \sum_{j=1}^{K}\left[\frac{\delta_{0n}}{v}Q_{j,stat}\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}D_j} + \frac{Q_{d,j}\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}D_j}\, f_{j,n}}{Lv}\right] \tag{11}$$

Where by the properties of the Dirac delta, $\delta_{0n} = 1$ if $n = 0$, and $\delta_{0n} = 0$ if $n \neq 0$. In the other hand, $f_{j,n}$ is the Fourier coefficient of the function $f_j(x)$ defined by $f_{j,n} = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f_j(x)\mathrm{e}^{-2\pi \mathrm{i} n\frac{x}{L}}dx$.

Now, the Fourier series expansion of equation 9 is performed. This comes to replacing equation 10 and 11, into equation 9, and then doing the expansion of the derivatives with respect to x accordingly. From these operations equation 9 becomes,

$$\begin{array}{l} \kappa SG\left(\frac{\mathrm{i}2\pi n}{L} - \frac{\mathrm{i}\omega}{v}\right)q_n = -\kappa SG\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)^2 p_n + \rho S\omega^2 p_n + \frac{\hat{R}_n(\omega)}{L} - \left(\frac{\delta_{0n}Q^S(\omega)}{v} + \frac{\hat{Q}_n^D(\omega)}{v}\right) \\ \kappa SG\left(\frac{\mathrm{i}\omega}{v} - \frac{\mathrm{i}2\pi n}{L}\right)p_n = -EI\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)^2 q_n - \left(\kappa SG - \rho I\omega^2\right)q_n \end{array} \tag{12}$$

With,

$$\begin{cases} \hat{R}_n(\omega) = \sum_{p=0}^{m-1} \hat{R}_p(\omega)\mathrm{e}^{\mathrm{i}\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)x_p} \\ Q^S(\omega) = \sum_{j=1}^{K} Q_{j,stat}\, \mathrm{e}^{-\mathrm{i}\frac{\omega}{v}D_j} \\ \hat{Q}_n^D(\omega) = \sum_{j=1}^{K} Q_{d,j}\, \mathrm{e}^{-\mathrm{i}\frac{\omega}{v}D_j} f_{j,n} \end{cases}$$

By isolating q_n from the second equation of system 12 and reinjecting it into 11, we have that the Fourier coefficients for $\Phi(x,\omega)$ and $\Psi(x,\omega)$ become respectively,

$$\begin{array}{l} p_n = \tilde{p}_n\hat{R}_n(\omega) - \left(\delta_{0n}\frac{\tilde{p}_0 L}{v}Q^S(\omega) + \frac{\tilde{p}_n L}{v}\hat{Q}_n^D(\omega)\right) \\ q_n = \tilde{q}_n\hat{R}_n(\omega) - \left(\delta_{0n}\frac{\tilde{q}_0 L}{v}Q^S(\omega) + \frac{\tilde{q}_n L}{v}\hat{Q}_n^D(\omega)\right) \end{array} \tag{13}$$

$\tilde{p}_n$, $\tilde{q}_n$, and z_n can be calculated through the following expressions according to [4],

$$\begin{cases} \tilde{p}_n = \frac{\kappa SG - \rho I\omega^2 + EI\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)^2}{Lz_n} \\ \tilde{q}_n = -\frac{i\kappa SG\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)}{Lz_n} \\ z_n = \kappa SGEI\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)^4 - \rho IS\omega^2(\kappa G + E)\left(\frac{\omega}{v} - \frac{2\pi n}{L}\right)^2 + \rho S\omega^2\left(\rho I\omega^2 - \kappa SG\right) \end{cases}$$

Replacing equation 13 into 10, we obtain the following relations,

$$\begin{array}{l} \Psi(x,\omega) = \hat{R}_n(\omega)\sum_{n=-\infty}^{\infty}\tilde{p}_n\mathrm{e}^{\mathrm{i}2\pi n\frac{x}{L}} - \left(\frac{\tilde{p}_0 L}{v}Q^S(\omega) + \sum_{n=-\infty}^{\infty}\frac{\hat{Q}_n^D L}{v}\tilde{p}_n\mathrm{e}^{\mathrm{i}2\pi n\frac{x}{L}}\right) \\ \Phi(x,\omega) = \hat{R}_n(\omega)\sum_{n=-\infty}^{\infty}\tilde{q}_n\mathrm{e}^{\mathrm{i}2\pi n\frac{x}{L}} - \left(\frac{\tilde{q}_0 L}{v}Q^S(\omega) + \sum_{n=-\infty}^{\infty}\frac{\hat{Q}_n^D L}{v}\tilde{q}_n\mathrm{e}^{\mathrm{i}2\pi n\frac{x}{L}}\right) \end{array} \tag{14}$$

In order to express equation 14 in a more compact and intuitive way, let's define,

$$\begin{array}{l} \eta(x,\omega) = \sum_{n=-\infty}^{\infty}\tilde{p}_n\mathrm{e}^{\mathrm{i}\frac{2\pi nx}{L}} \\ \gamma(x,\omega) = \sum_{n=-\infty}^{\infty}\tilde{q}_n\mathrm{e}^{\mathrm{i}\frac{2\pi nx}{L}} \end{array} \tag{15}$$

Thus, we can finally obtain an expression to define the vertical displacement at position x of the beam (rail), and at frequency ω. Introducing the first equation of system 15 into the first equation of 14, and considering the relation defined in 8, we get,

$$\begin{aligned}\hat{w}_r(x,\omega) = \sum_{p=0}^{m-1} \hat{R}_p(\omega) \mathrm{e}^{\mathrm{i}\frac{\omega}{v}(x_p - x)} \eta(x - x_p, \omega) - \sum_{j=1}^{K} \Bigg(\frac{\tilde{p}_0 L}{v} Q_{j,stat} \mathrm{e}^{-\mathrm{i}\frac{\omega}{v} D_j} \\ + \sum_{n=-\infty}^{\infty} \tilde{p}_n \mathrm{e}^{\mathrm{i}\frac{2\pi n}{L} x} \frac{Q_{d,j} L}{v} \mathrm{e}^{-\mathrm{i}\frac{\omega}{v} D_j} f_{n,j} \Bigg) \mathrm{e}^{-\mathrm{i}\frac{\omega}{v} x}\end{aligned} \tag{16}$$

The equation 16 can be rewritten in the following form in order to be compare to the beam displacement expression that can be found in other texts such as [1,4]

$$\hat{w}_r(x,\omega) = \sum_{p=0}^{m-1} \hat{R}_p(\omega) \mathrm{e}^{\mathrm{i}\frac{\omega}{v}(x_p - x)} \eta(x - x_p, \omega) - \left(\eta(0,\omega) \mathcal{Q}_S(\omega) + \tilde{\mathcal{Q}}_D^{\star}(\omega) \right) \mathrm{e}^{-\mathrm{i}\frac{\omega}{v} x} \tag{17}$$

With,

$$\begin{cases} \forall x \in [0;L], \eta(x,\omega) = \dfrac{\mathrm{e}^{\mathrm{i}\frac{\omega x}{v}}}{2EI\left(\lambda_1^2 + \lambda_2^2\right)} \left[\dfrac{C_1}{\lambda_1} \dfrac{\sin \lambda_1 (L - x) + \mathrm{e}^{-\mathrm{i}\frac{\omega L}{v}} \sin \lambda_1 x}{\cos L\lambda_1 - \cos \frac{\omega L}{v}} \right. \\ \left. - \dfrac{C_2}{\lambda_2} \dfrac{\sinh \lambda_2 (L - x) + \mathrm{e}^{-\mathrm{i}\frac{\omega L}{v}} \sinh \lambda_2 x}{\cosh L\lambda_2 - \cos \frac{\omega L}{v}} \right] \\ \mathcal{Q}_S(\omega) = \dfrac{\tilde{p}_0 L}{v} \eta(0,\omega)^{-1} \displaystyle\sum_{j=1}^{K} Q_{j,stat} \mathrm{e}^{-\mathrm{i}\omega \frac{D_j}{v}} \\ \tilde{\mathcal{Q}}_D^{\star}(\omega) = \displaystyle\sum_{j=1}^{K} \sum_{n=-\infty}^{\infty} \tilde{p}_n \mathrm{e}^{\mathrm{i}\frac{2\pi n}{L} x} \dfrac{Q_{d,j} L}{v} \mathrm{e}^{-\mathrm{i}\frac{\omega}{v} D_j} f_{n,j} \\ C_{1,2} = 1 - \dfrac{\rho I \omega^2 \mp EI \lambda_{1,2}^2}{\kappa S G} \\ \tilde{p}_0 L = \dfrac{\kappa S G - \rho I \omega^2 + EI \frac{\omega^2}{v^2}}{\kappa S G \left(EI \frac{\omega^4}{v^4} - \rho S \omega^2 \right) - \rho S I \left(\kappa G + E - \rho v^2 \right) \frac{\omega^4}{v^2}} \end{cases}$$

Equation 17 is the result of the periodic property inherent in the system of interest and the dynamic equation of the Timoshenko beam.

The term $\tilde{\mathcal{Q}}_D^{\star}(\omega)$ implicitly contains the existing interaction between the beam's behaviour and the dynamic loading. Having into account that the problem treated here comes to a direct problem solution, the function $Q_{j,dyn}(x)$ is given and expression 16 can be solved. Section 3 presents few study case when imposing a function to $Q_{j,dyn}(x)$ in order to asset the railway track's behaviour subjected to static and dynamic loading.

To propose a general form, let's set $x = x_q$. In this case, considering that $\eta(x, \omega)$ is L-periodic, then $\eta_{(q,p)}(\omega) = \eta_{(q,m+p)}(\omega) = \eta(x_p - x_q, \omega)$. Furthermore, we highlight the relation between the displacement for the general formulation, where $\hat{w}_q(\omega) = \hat{w}_r(x_q, \omega)\mathrm{e}^{\mathrm{i}\frac{\omega}{v}x_q}$ .Thus, equation 17 consequently becomes,

$$\hat{w}_q(\omega) = \sum_{p=0}^{m-1} \eta_{q,p}\hat{R}_p(\omega) - (\eta_0(\omega)\mathcal{Q}_S(\omega) + \tilde{\mathcal{Q}}_D^{\star}) \tag{18}$$

Or written in a matricial form,

$$\underline{\underline{\mathbf{C}_p}}\,\underline{\hat{\mathbf{R}}} = \underline{\hat{\mathbf{w}}} + (\eta_0\mathcal{Q}_S\underline{\mathbf{1}} + \underline{\tilde{\mathcal{Q}}_D^{\star}}) \tag{19}$$

In equation 19, we have a relationship between the support reaction force and the vertical displacement of the beam. However, we do not yet know any of these two values. To do this, it is necessary to introduce an additional relation that will come from the equation of dynamic stiffness of the support motif. The discussion to come will therefore be intended to address the various existing formulations to have a relationship between the vertical displacement of the supports and its reaction force, in order to calculate the response of the railway track subjected to loads.

2.3 Direct reponses computation

For cases where we want to calculate the track response for a configuration with only a few dozen supports and where the reaction forces that provide the supports are linearly related to displacement, it is preferable to use the direct formulation proposed here. This formulation is based on the support stiffness equation, which in its matrix form is written as follows:

$$\underline{\hat{\mathbf{w}}} = -\underline{\underline{\mathbf{D}}}\,\underline{\hat{\mathbf{R}}} \tag{20}$$

Equation 20 is the last relation we need to calculate the track response. In equation 20, the unknown vector is the vector of reaction forces $\underline{\hat{\mathbf{R}}}$. However, the support stiffness equation can also be formulated so that the unknown vector is the one containing the displacements of the supports, as follows:

$$\underline{\hat{\mathbf{R}}} = -\underline{\underline{\mathbf{K}}}\,\underline{\hat{\mathbf{w}}} \tag{21}$$

For the force formulation described by equation 20, $\underline{\underline{\mathbf{D}}}$ contains the inverses of the dynamic stiffnesses of the supports of the motif in its diagonal. This can be expressed as follows:

$$\underline{\underline{\mathbf{D}}} = \begin{pmatrix} k_{s,0}^{-1} & \cdots & \cdots & \mathbf{0} \\ \vdots & k_{s,1}^{-1} & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & k_{s,m-1}^{-1} \end{pmatrix}$$

By replacing equation 20 into equation 19,

$$\underline{\hat{\mathbf{R}}} = \underline{\underline{\mathbf{A}}}^{-1}(\underline{\mathbf{1}}\eta_0 \mathcal{Q}_S + \underline{\tilde{\mathcal{Q}}_D^{\star}}) \tag{22}$$

With $\underline{\underline{\mathbf{A}}} = \left(\underline{\underline{\mathbf{C}_p}} + \underline{\underline{\mathbf{D}}}\right)$.

As for the displacement formulation described by equation 21, $\underline{\underline{\mathbf{K}}}$ is the matrix containing the dynamic stiffnesses of the supports of the motif L in its diagonal. It has the following form:

$$\underline{\underline{\mathbf{K}}} = \begin{pmatrix} k_{s,0} & \cdots & \cdots & \mathbf{0} \\ \vdots & k_{s,1} & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & k_{s,m-1} \end{pmatrix}$$

By replacing equation 21 into equation 19,

$$\underline{\hat{\mathbf{w}}} = -\underline{\underline{\mathbf{B}}}^{-1}(\underline{\mathbf{1}}\eta_0 \mathcal{Q}_S + \underline{\tilde{\mathcal{Q}}_D^{\star}}) \tag{23}$$

With $\underline{\underline{\mathbf{B}}} = \left(\underline{\underline{\mathbf{C}_p}}\,\underline{\underline{\mathbf{K}}} + \underline{\underline{\mathbf{I}}}\right)$.

For the formulations presented above, at each frequency, the matrices $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{B}}}$ of dimensions $m \times m$ must be calculated, and then inverted. The formulation of equation 22 is preferable when we want to know the impact of a stiffer support in a motif. On the contrary, to know the effect of supports with low (or even zero) stiffness in a motif, it is preferable to use equation 23.

3 Study case

In the present section a dynamic loading function is going to be imposed. To simplify expressions and make the analysis easier to perform, we are going to treat the case where the structure is going to be loaded by the passage of only one wheel ($D_j = 0$ and $K = 1$). Applying this previously simplification, the equation describing the railway's track vertical displacement comes to,

$$\hat{w}_q(x_q,\omega) = \sum_{p=0}^{m-1} \hat{R}_p(\omega)\mathrm{e}^{\mathrm{i}\frac{\omega}{v}(x_p)}\eta(x_q - x_p,\omega) - \left(\frac{\tilde{p}_0 L}{v} Q_{stat} + \sum_{n=-\infty}^{\infty} \frac{\tilde{p}_n L}{v}\mathrm{e}^{\mathrm{i}\frac{2\pi n x_q}{L}} f_n\right) \tag{24}$$

The mechanical parameters describing the Timoshenko's beam beaviour are presented in table 1.

Parameter	Notation	Value
Rail mass	ρS	$60\ \mathrm{kg}\cdot\mathrm{m}^{-1}$
Rail flexion stiffness	EI	$6.38\mathrm{MN}\cdot\mathrm{m}^2$
Timoshenko's ratio	κ	0.4
Shear modulus	G	8.077GPa
Train's speed	v	$37\ \mathrm{m}\cdot\mathrm{s}^{-1}$
Block's mass	M	100 kg
Suppports spacing	l	0.6 m
Railpad stiffness	k_r	$192\mathrm{MN}\cdot\mathrm{m}^{-1}$
Railpad damping	ξ_r	$1.97\mathrm{MN}\cdot\mathrm{s}\cdot\mathrm{m}^{-1}$
Under-sleeper stiffness	k_f	$26.4\mathrm{MN}\cdot\mathrm{m}^{-1}$
Under-sleeper damping	ξ_f	$0.17\mathrm{MN}\cdot\mathrm{s}\cdot\mathrm{m}^{-1}$

Table 1: Parameters used for railway's track responses computation.

For this example, the dynamic loading $Q_{dyn}(x)$ will take the form of a Heaviside step function, defined as follows,

$$Q_{dyn}(x) = \mathcal{H}(x) = \begin{cases} A & \text{if } x \in [--a, a] \\ 0 & \text{otherwise} \end{cases}$$

Being A the dynamic loading amplitude. When performing a Fourier transform over this last function, it takes the following form,

$$f_n = \begin{cases} \frac{\sin \pi n \frac{a}{L}}{\pi n}, & n \neq 0 \\ \frac{a}{L}, & n = 0 \end{cases}$$

Thus, the expression to compute the railway track vertical displacement becomes,

$$\hat{w}_q(x_q,\omega) = \sum_{p=0}^{m-1} \hat{R}_p(\omega)\mathrm{e}^{\mathrm{i}\frac{\omega}{v}(x_p)}\eta(x_q - x_p,\omega) - \left(\frac{\tilde{p}_0 L}{v} Q_{stat} + \sum_{n=-\infty}^{\infty} \frac{\tilde{p}_n L}{v}\mathrm{e}^{\mathrm{i}\frac{2\pi n x_q}{L}} \frac{1}{\pi n}\sin(\frac{\pi n a}{L})\right) \tag{25}$$

From this point, we establish that the section of ballastless railway track to be studied in this case will consist of $m = 40$ equally spaced supports with the mechanical properties outlined in Table 1.

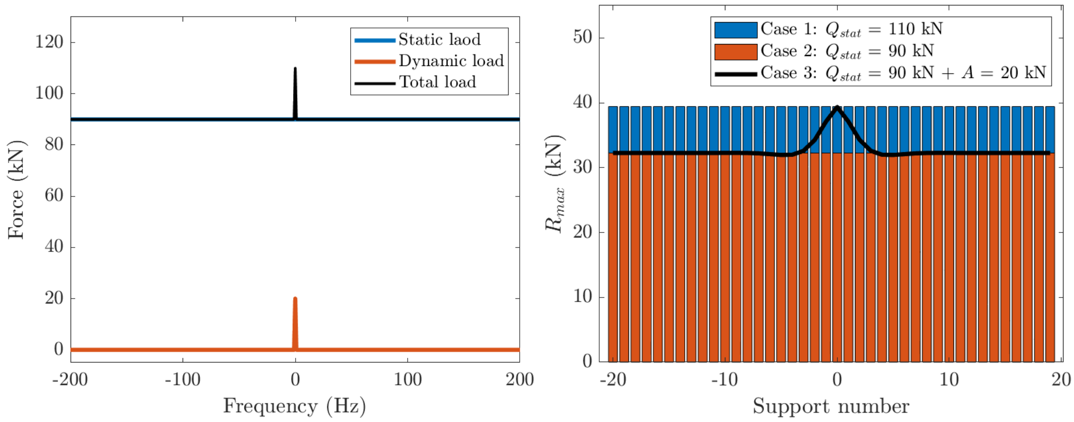

Figure 2: Loading conditions used to compute the railway's track response (on the left). Maximum reaction force experienced by each support n over the imposed loading conditions (on the right).

Three different cases will be considered. The first two cases represent the railway track statically loaded by a force $Q_{\text{stat}} = 90$ N/m for the first case and $Q_{\text{stat}} = 110$ N/m for the second case. This loading condition models a constant force moving along the structure at a constant speed of $v = 37$ m/s. The third case to be studied will consider the same railway track loaded by a force traveling at the same speed as before, having a static component where $Q_{\text{stat}} = 90$ N/m and a dynamic component where the support $n = 0$ will experience a load amplitude $A = 20$ kN when the moving load passes over it.

Figure 2 illustrates the loading conditions imposed on the structure, along with the maximum reaction forces experienced by each of the supports for the three different cases studied. In case 3, it can be observed that the supports located far from the zone where the dynamic shock occurs have the same reaction force as the supports studied in case 1. This aligns with our expectations, as the dynamic loading at a distant location would already be absorbed by the supports located nearby to the dynamic loading zone. Consequently, support $n = 0$ from case 3 exhibits the same response as the supports in case 2 since they would be subjected to the same total load. Finally, for the supports located at $n_0 \pm 5$, they represent a transition zone where the supports are initially loaded, then perturbed, and after the perturbation dissipates, they return to their equilibrium state.

4 Concluding remarks

This article utilizes Timoshenko beam theory on a condition that considers a periodic structure to calculate its response to a load. Incorporating the contribution of dynamic loads into this analytical model of railway track presents significant interest when aiming to perform pre-dimensioning calculations of the track or to reliably and quickly assess how changes in various parameters may affect track behavior. By representing realistic force formulations, this approach enables exploration of maintenance strategies by assessing the impact of various parameters, such as support spacing in overloaded areas. Moreover, it offers insights into the track's behavior under precise dynamic loading functions, facilitating the study of how the structure responds to simulated track defects or wheel irregularities.

References

[1] B. Claudet, D. Duhamel, G. Foret, T. Hoang, J.L. Pochet, F. Sabatier, ”A Comparison of Beam Models for the Dynmaics of Railway Tracks on a Non-Uniform Viscoelastic Fondation”, ICSV27, Ostrava, Czech Republic, Jul 2021.

[2] J.F. Hamet, ”Railway Noise: Use of the Timoshenko Model in Railway Vi- bration Studies”, Acta Acustica united with Acustica, 54–62, 1999.

[3] M.A. Heckel, ”Coupled waves on a periodically supported Timoshenko beam”, Journal of Sound and Vibration, 252, 849-882, 2002.

[4] T. Hoang, D. Duhamel, and G. Foret, “Dynamical response of a Timoshenko beams on periodical nonlinear supports subjected to moving forces”, Engineering Structures, 176, 673–680, 2018.

[5] T. Hoang, ”Comportement mécanique et dégradation d'une voie ferrée en tunnel”, PhD Thesis, Université Paris-Est, 2011.

[6] X. Zhang, D. Thompson, X. Sheng, “Differences between Euler-Bernoulli and Timoshenko beam formulations for calculating the effects of moving loads on a periodically supported beam”. Journal of Sound and Vibration, 481, 115432, 2020.