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Advanced Continuation Methods for Limit Load and Shear Strength Reduction Methods

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Abstract

This contribution is focused on numerical determination of factors of safety within slope stability assessment and other applications. The limit load and the shear strength reduction methods are considered, combined with the Mohr-Coulomb plasticity and the finite element method. To suppress spurious numerical oscillations observed for nonassociated plastic models, Davis' modifications of the limit load and the shear strength reduction methods are considered. To suppress numerical overestimation of the factors of safety, a special continuation technique is suggested and its convergence with respect to a spatial discretization parameter is discussed. The paper also contains many useful details to this problematic which are explained on an algebraic level to be easily understandable.

Keywords: stability assessment, factor of safety, computational plasticity, finite element method, Newton-like methods, continuation techniques.

1 Introduction

Factors of safety (FoS) and failure zones of structures are investigated within stability analysis of structures in civil engineering and geotechnics by various analytical or numerical methods. We focus on the *limit load* (LL) and *shear strength reduction* (SSR) methods, their connections with elasto-plasticity and displacement-type finite elements. Briefly speaking, these methods are based on a parametrization of the elasto-plastic problem by a scalar factor. A critical value of the factor defines FoS. In the LL method, external forces are enlarged by the factor while in the SSR method, strength parameters are reduced. Although such approaches are well-known, several numerical difficulties may appear.

First, spurious numerical oscillations are sometimes observed for nonassociated plastic models [1,2] and they cause the nonuniqueness of FoS or a failure mechanism. To prevent this drawback, we build on Davis' modifications of the LL and SSR methods [1, 3–5]. The idea is to approximate the nonassociated plasticity with a sequence of *associated* plastic models. The approximation is straightforward in the LL method unlike the SSR method where an iterative procedure is recommended.

Second, FoS may be overestimated or strongly dependent on mesh density. This mainly occurs if the simplest linear finite elements are considered. Therefore, higher-order finite elements (e.g. quadrilateral elements) are recommended in many commercial codes. However, such a treatment can be too slow for 3D problems. A more rigorous approach builds on mesh-independent optimization problems related to the (modified) LL and SSR methods. The optimization framework for the LL method is well-known for many decades and is called the *limit analysis* (LA) problem [6, 7]. In engineering literature, we distinguish lower and upper bound limit analysis theorems defining lower and upper bounds of FoS, see [3] and the references therein. Recently, the optimization approach was extended to the modified SSR method [2, 5]. There exist convergence results with respect to the discretization parameters which enable us to suppress the overestimation of FoS [7–9]. In addition, various mesh adaptive strategies were developed [2, 3, 5, 10].

In this contribution, the modified LL and the SSR methods are introduced for the Mohr-Coulomb elastic-perfectly plastic problem. Then, these methods are investigated on an algebraic level in order to transparently define convenient numerical algorithms. We build on continuation techniques which are closely related to regularization of the optimization framework. We also present a methodology enabling to prevent the overestimation of FoS.

2 Discretized elastic-plastic problem and its parametrizations

Elasto-plastic problems after time and space discretization are often defined in terms of displacements and have the following scheme:

$$\text{find } \boldsymbol{u}^* \in V_h: \quad \int_\Omega T\left(\boldsymbol{\varepsilon}(\boldsymbol{u}^*)\right) : \boldsymbol{\varepsilon}(\boldsymbol{u})dx = b(\boldsymbol{u}) \quad \forall \boldsymbol{u} \in V_h, \tag{1}$$

where V_h is a finite-dimensional space of admissible displacements, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, that is, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$, the functional b represents the work of external forces (e.g., a combination of volume and surface forces) and

$T\colon \boldsymbol{\varepsilon} \mapsto \boldsymbol{\sigma}$ is a nonlinear constitutive operator which maps the strain tensor into the Cauchy stress $\boldsymbol{\sigma}$. If an elastic-perfectly plastic model is considered then this mapping can be defined as follows [11, 12]:

Given $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p \in \mathbb{R}^{3\times 3}_{sym}$, *find* $\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}_{sym}$, $\boldsymbol{\mu} \in \mathbb{R}^{3\times 3}_{sym}$ *and* $\gamma \in \mathbb{R}$ *satisfying:*

$$\left.\begin{array}{c} \boldsymbol{\sigma} = \mathbb{D}_e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p - \gamma\boldsymbol{\nu}), \quad \boldsymbol{\nu} \in \partial g(\boldsymbol{\sigma}), \\ \gamma \geq 0, \quad f(\boldsymbol{\sigma}) \leq 0, \quad \gamma f(\boldsymbol{\sigma}) = 0. \end{array}\right\} \tag{2}$$

Here, $\boldsymbol{\varepsilon}^p$ denotes the plastic strain tensor obtained in the previous time step, γ is the plastic multiplier, $\mathbb{D}_e$ is the fourth-order elastic tensor representing the Hooke's law, f, g are given convex functions representing a yield criterion and a plastic potential, respectively, $\partial g(\boldsymbol{\sigma})$ denotes the subdifferential of g at $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$ is an element belonging to $\partial g(\boldsymbol{\sigma})$. If g is differentiable at $\boldsymbol{\sigma}$ then $\partial g(\boldsymbol{\sigma})$ is a singleton and one can simply write $\boldsymbol{\nu} = \partial g(\boldsymbol{\sigma})/\partial\boldsymbol{\sigma}$ as is usual in engineering literature. The operator $T : \boldsymbol{\varepsilon} \mapsto \boldsymbol{\sigma}$ defined by (2) is not differentiable everywhere but its generalized derivative exists and will be denoted as T^o. Its knowledge is important for Newton-like solvers of the elasto-plastic problem.

The limit load (LL) method means that the problem (1) is parametrized by a scalar factor $t \geq 0$ which multiplies the load vector b. The factor of safety (FoS) is defined as a maximal value of t for which the parametrized problem has a solution. So we see that the LL method is very universal and can be applied for various elasto-plastic models.

To introduce the shear strength reduction (SSR) method and Davis' modifications of the LL and SSR methods, we shall consider an isotropic material and the Mohr-Coulomb (MC) model, for the sake of simplicity. Then, we have 5 material parameters: the bulk modulus, K, the shear modulus, G, the (effective) cohesion, c, the (effective) friction angle, ϕ, and the dilatancy angle, ψ. Further,

$$\begin{aligned} \mathbb{D}_e\boldsymbol{\varepsilon} &= (K - 2G/3)(\operatorname{tr}\boldsymbol{\varepsilon})\boldsymbol{I} + 2G\boldsymbol{\varepsilon}, \\ f(\boldsymbol{\sigma}) &= (1+\sin\phi)\sigma_1 - (1-\sin\phi)\sigma_3 - 2c\cos\phi, \\ g(\boldsymbol{\sigma}) &= (1+\sin\psi)\sigma_1 - (1-\sin\psi)\sigma_3 - 2c\cos\psi, \end{aligned}$$

where $\operatorname{tr}\boldsymbol{\varepsilon}$ is the trace of $\boldsymbol{\varepsilon}$, $\boldsymbol{I}$ denotes the unit second-order tensor and σ_1, σ_3 are the maximal and minimal principle stresses, respectively. (We use the standard mechanical sign convention where the tension has a positive sign). The construction of the operators T and T^o for the Mohr-Coulomb model can be found e.g. in [11, 12].

Within the SSR method, the strength parameters c, ϕ and ψ are reduced using a scalar factor $\lambda > 0$:

$$c_\lambda := \frac{c}{\lambda}, \quad \phi_\lambda := \arctan\frac{\tan\phi}{\lambda}, \quad \psi_\lambda := \arctan\frac{\tan\psi}{\lambda}. \tag{3}$$

The corresponding FoS is defined as a maximal value of λ for which the parametrized problem (1) has a solution. The SSR method is conventional in slope stability but not so universal as the LL method.

If $\psi = \phi$ then $f = g$ and we talk about the *associated* MC model with clear mathematical theory based on an optimization background presented below. However, in geotechnics, the *non-associated* MC model with $0 \leq \psi < \phi$ is more usual which can lead to the numerical difficulties discussed in Section 1.

Davis' modifications of the LL and SSR methods are based on approximations of the non-associated MC model with associated ones. Such an approximation was originally suggested for the LL method (see [3] and the references therein) where the strength parameters c, ϕ and ψ are modified as follows:

$$\bar{c} = \beta c, \quad \bar{\phi} = \bar{\psi} = \arctan(\beta \tan \phi), \quad \beta := \frac{\cos \psi \cos \phi}{1 - \sin \psi \sin \phi}. \tag{4}$$

The LL method is then applied for the modified values $\bar{c}$, $\bar{\phi} = \bar{\psi}$ instead of the original values c, ϕ and ψ. Let us note that the values of $\bar{c}$ and $\bar{\phi}$ are usually lower than c and ϕ.

The Davis approach can be extended to the SSR method [1,4]. In particular, the strength parameters c and ϕ are modified for any factor $\lambda > 0$. A general scheme of the modified SSR method was introduced in [5]:

$$\tilde{c}_\lambda := \frac{c}{q(\lambda; \phi, \psi)}, \quad \tan \tilde{\phi}_\lambda = \tan \tilde{\psi}_\lambda := \frac{\tan \phi}{q(\lambda; \phi, \psi)}, \tag{5}$$

where q is a scalar function satisfying:

- q is positive and continuous for any $\lambda > 0$ and any ϕ, ψ such that $0 \leq \psi \leq \phi$;
- q is increasing with respect to the variable $\lambda \geq 0$;
- q is non-increasing with respect to the variable $\psi \in [0, \phi]$;
- $q(\lambda; \phi, \psi) \geq \lambda$ for any $\lambda \geq 0$ and any ϕ, ψ such that $0 \leq \psi \leq \phi$;
- if $\psi = \phi$ then $q(\lambda; \phi, \psi) = \lambda$.

The following three examples of the function q with the above-mentioned properties were introduced and analyzed in [5]:

$$q_A(\lambda; \phi, \psi) = \lambda \frac{1 - \sin \psi \sin \phi}{\cos \psi \cos \phi},$$

$$q_B(\lambda; \phi, \psi) = \lambda \frac{1 - \sin \psi_\lambda \sin \phi_\lambda}{\cos \psi_\lambda \cos \phi_\lambda}, \quad \phi_\lambda = \arctan \frac{\tan \phi'}{\lambda}, \quad \psi_\lambda = \arctan \frac{\tan \psi'}{\lambda},$$

$$q_C(\lambda; \phi, \psi) = \begin{cases} \lambda \frac{1 - \sin \psi \sin \phi_\lambda}{\cos \psi \cos \phi_\lambda}, & \text{if } \phi_\lambda \geq \psi, \\ \lambda, & \text{if } \phi_\lambda \leq \psi, \end{cases} \quad \phi_\lambda = \arctan \frac{\tan \phi'}{\lambda}.$$

These functions correspond to the Davis A, B and C approaches suggested in [4].

With respect to the Davis modifications of the LL and SSR problems, we shall consider from now on only the associated model with $f = g$ in (2). Under this assumption, the operator $T : \boldsymbol{\varepsilon} \mapsto \boldsymbol{\sigma}$ defined by (2) has the following potential [5]:

$$\Psi_T(\boldsymbol{\varepsilon}) = \max_{\boldsymbol{\tau},\, f(\boldsymbol{\tau})\leq 0} \left[\boldsymbol{\tau} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \frac{1}{2}\mathbb{D}_e^{-1}\boldsymbol{\tau} : \boldsymbol{\tau}\right]. \tag{6}$$

Here, the Cauchy stress $\boldsymbol{\sigma}$ maximizes the functional on the right-hand side subject to the constraint $f(\boldsymbol{\tau}) \leq 0$. In addition, $\frac{\partial \Psi_T(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = T(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma}$. The potential enables us to introduce an optimization definition of the elasto-plastic problem, see Section 3.

Finally, we also need to consider the following local dissipation function related to the elasto-plastic constitutive problem:

$$\Psi_\infty(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\tau},\, f(\boldsymbol{\tau})\leq 0} \left[\boldsymbol{\tau} : \boldsymbol{\varepsilon}\right].$$

The supremum is used here instead of the maximum because the case $\Psi_\infty(\boldsymbol{\varepsilon}) = +\infty$ can occur for some $\boldsymbol{\varepsilon}$. The dissipation potential enables us to introduce an optimization definition of the LL and SSR methods, see the next sections.

3 The modified LL method in an algebraic form

From now on, we shall work only with the algebraic setting of the problem (1), for the sake of simplicity:

$$\text{find } u^* \in \mathbb{R}^n: \quad F(u^*) = b, \qquad F\colon \mathbb{R}^n \to \mathbb{R}^n,\ b \in \mathbb{R}^n, \tag{7}$$

where u^* is the unknown displacement vector and the nonlinear function F can be assembled from the integral

$$\int_\Omega T\left(\boldsymbol{\varepsilon}(\boldsymbol{u})\right) : \boldsymbol{\varepsilon}(\boldsymbol{v})dx.$$

Similarly, we introduce other algebraic functions $F^o : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathcal{I} : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{I}_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ corresponding to the above-mentioned constitutive operators T^o, Ψ_T and Ψ_∞, respectively. That is, $\nabla\mathcal{I}(u) = F(u)$ and $\nabla F(u) = F^o(u)$ if F is differentiable at u. From the mathematical point of view, one can expect that the function F is Lipschitz continuous and strongly semismooth. Therefore, the following semismooth Newton method can be used and has local quadratic convergence:

$$u^{k+1} = u^k + [F^o(u^k)]^{-1}(b - F(u^k)),\quad k = 0, 1, \ldots, \quad u^0 - \text{ given.}$$

Further, using the function $\mathcal{I}$, one can introduce the following minimization problem, which is equivalent to (7):

$$\mathcal{J}(u^*) \leq \mathcal{J}(v)\ \ \forall v \in \mathbb{R}^n, \quad \text{where} \quad \mathcal{J}(v) = \mathcal{I}(v) - b^\top v. \tag{8}$$

From the mathematical point of view, $\mathcal{I}$ is a convex function having a linear growth at infinity. It implies that the function $\mathcal{J}$ has a minimum if the load vector b is sufficiently small.

The modified LL method provides more accurate information about the solvability of (7) or (8). It suffices to parametrize the problem (7) using a scalar load factor $t \geq 0$:

$$\text{for any } t \geq 0, \text{ find } \hat{u}(t) \in \mathbb{R}^n : \quad F(\hat{u}(t)) = tb. \tag{9}$$

The limit load factor will be denoted as t^* and is naturally defined by

$$t^* \;=\; \text{maximum of } \; t \geq 0 \; \text{ such that the solution } \hat{u}(t) \text{ of (9) exists.} \tag{10}$$

More precisely, "supremum" instead of "maximum" should be used in (10) because the system (9) often does not have a solution for $t = t^*$. The case $t^* = +\infty$ can also occur in some extreme cases, for example if the friction and dilatancy angles are greater than 45 degrees. It is worth noticing that the original system $F(u^*) = b$ has a solution if $t^* > 1$.

The limit load factor t^* can also be defined by the *limit analysis* problem [6, 7]:

$$t^* = \inf_{\substack{v \in \mathbb{R}^n \\ b^\top v = 1}} \mathcal{I}_\infty(v) = \min_{\substack{v \in \mathcal{C} \\ b^\top v = 1}} \mathcal{I}_\infty(v), \tag{11}$$

where $\mathcal{I}_\infty$ is the *dissipation function* and $\mathcal{C} = \{v \in \mathbb{R}^n \mid \mathcal{I}_\infty(v) < +\infty\}$ is the corresponding feasible set. This "hidden" constraint depends on a chosen yield criterion. From the optimization definition (11), we see that any feasible v defines an upper bound of t^*. Therefore, we talk about the *upper bound theorem of limit analysis* in engineering practice.

These two definitions of t^* are equivalent and for purposes of this paper it is useful to briefly derive it. For more details, we refer to [8–10, 13–15]. We start with the following relationship between the functions $\mathcal{I}_\infty$ and $\mathcal{I}$:

$$\mathcal{I}_\infty(v) := \lim_{\omega \to +\infty} \mathcal{I}_\omega(v) \quad \forall v \in \mathbb{R}^n, \quad \text{where} \quad \mathcal{I}_\omega(v) = \frac{1}{\omega}\mathcal{I}(\omega v), \; \omega > 0. \tag{12}$$

Hence, we see that $\mathcal{I}_\infty$ can be regularized by the smooth function $\mathcal{I}_\omega$ which is finite-valued everywhere and can be easily derived from the original function $\mathcal{I}$. For any regularization parameter $\omega > 0$, one can define the following auxiliary problem:

$$\mathcal{I}_\omega(v(\omega)) = \min_{\substack{v \in \mathbb{R}^n \\ b^\top v = 1}} \mathcal{I}_\omega(v). \tag{13}$$

This problem can be transformed by substitution into the following one:

$$\mathcal{I}(\bar{u}(\omega)) = \min_{\substack{v \in \mathbb{R}^n \\ b^\top v = \omega}} \mathcal{I}(v), \tag{14}$$

where $v(\omega) = \bar{u}(\omega)/\omega$. Analyzing the optimality conditions for (14) we arrive at the following saddle-point system for given $\omega > 0$:

$$\text{find } \bar{u}(\omega) \in \mathbb{R}^n, \; t(\omega) \geq 0 : \qquad F(\bar{u}(\omega)) = t(\omega)b, \;\; b^T \bar{u}(\omega) = \omega. \tag{15}$$

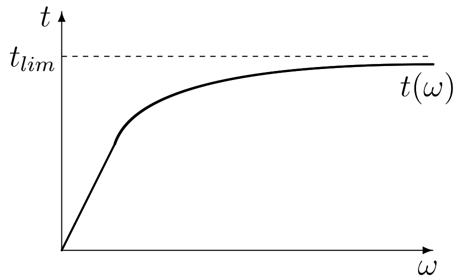

Figure 1: Control of the loading process through the work of external forces.

It holds that the system (15) has always a solution unlike the original system (9). Further, comparing the solutions of the systems (15) and (9), we see that

$$\bar{u}(\omega) = \hat{u}(t(\omega)).$$

Finally, it holds that

$$t(\omega) \leq t^*, \quad \lim_{\omega \to +\infty} t(\omega) = t^*$$

as is sketched in Figure 1.

We see that the additional variable ω represents a prescribed value of the work of external forces and that the loading process can be controlled indirectly through ω instead of the simplest increasing of the parameter t. One can introduce the following continuation method for the construction of the function $\omega \mapsto t(\omega)$:

1. Generate (adaptively) a sequence $0 < \omega_1 < \omega_2 < \ldots < \omega_N$.

2. For any $j = 1, 2, \ldots, N$, solve the system (15) and find its solution $\bar{u}(\omega_j) \in \mathbb{R}^n,\ t(\omega_j) \geq 0$.

3. Approximate t^* by $t(\omega_N)$ where ω_N is sufficiently large.

We complete this algorithm with the following remarks (recommendations):

- We start with a constant increment of ω. Its convenient value can be estimated e.g. using the external work $b^T u_{el}^*$ of the elastic solution satisfying $K_{el} u_{el}^* = b$.

- The increment of ω is enlarged (e.g. twice) if the corresponding increments of t are smaller than a prescribed tolerance (e.g. 10^{-3}).

- If the increments of ω are enlarged several times (e.g. 5 times) then the algorithm is terminated.

- It is natural to approximate t^* by $t(\omega_N)$. However, to prevent possible numerical overestimation of t^*, it is important to store the whole curve $\omega \mapsto t(\omega)$ as is explained in Section 5.

- Similar continuation techniques are also known from literature. For example, the loading process can be controlled by a displacement at a selected point (see e.g. [12]) or by the arc-length method [11, 16].

Finally, we introduce the Newton-like method with damping for solving the minimization problem (14) for a fixed ω:

$$u^{k+1} := u^k + \alpha_k s^k \qquad k = 0, 1, \dots, \quad u^0 - \text{ given, } b^\top u^0 = \omega, \tag{16}$$

$$s^k = \arg \min_{\substack{s \in \mathbb{R}^n \\ b^\top s = 0}} \left[\frac{1}{2} (F^o(u^k)s, s) + (F(u^k), s) \right], \tag{17}$$

$$\alpha_k = \arg \min_{\alpha \in [0,1]} \mathcal{I}(u^k + \alpha s^k). \tag{18}$$

The following remarks provide more details to this algorithm:

- This method can also be interpreted as a sequential quadratic programming. The quadratic functional in (17) approximates $\mathcal{I}$ using the Taylor expansion.

- The initial vector u^0 is achieved by extrapolation from previous two solutions $\bar{u}(\omega_{j-1})$ and $\bar{u}(\omega_{j-2})$ corresponding to smaller values of ω than the current one.

- The sequence $\{u^k\}_k$ generated by this algorithm satisfies $b^\top u^k = \omega$.

- The vector s^k is a descent direction of $\mathcal{I}$. The minimization problem (17) defining s^k can be transformed to the following system of linear equations:

  $$\text{find } s^k \in \mathbb{R}^n, \ t_k \geq 0 : \quad F^o(u^k)s^k = t_k b - F(u^k), \ \ b^T s^k = 0. \tag{19}$$

  The solution component t_k is used for approximation of $t(\omega)$. This system can be solved by superposition applied on $F^o(u^k)s^k = t_k b - F(u^k)$. In particular, we consider the following split of s^k within the superposition:

  $$s^k = v^k + (t_k - t_{0,k})w^k, \tag{20}$$

  where $t_{0,k}$ is given and

  $$F^o(u^k)v^k = t_{0,k} b - F(u^k), \qquad F^o(u^k)w^k = b. \tag{21}$$

  It is easy to verify that (s^k, t_k) solves (19) if (20)–(21) hold. Further, we see that two linearized systems of equations with the same stiffness matrix $F^o(u^k)$ are solved in each Newton's iteration. The additional parameter $t_{0,k}$ can be chosen arbitrary, for example, one can set $t_{0,k} = t_{k-1}$. The reason of this parameter is to reduce rounding errors within the computation of s^k because the matrix $F^o(u^k)$ can be ill-posed for higher ω.

- If we observe that the matrix $F^o(u^k)$ is almost singular then we replace it with its regularization in the form $(1-\beta)F^o(u^k) + \beta K_{el}$, where $\beta \in (0, 1)$ and K_{el} is the elastic stiffness matrix.

- The damping parameter α_k is important for global convergence of the algorithm because its presence guarantees that $\mathcal{I}(u^{k+1}) \leq \mathcal{I}(u^k)$ for any $k = 0, 1, \ldots$. Notice that it is not necessary to evaluate the function $\mathcal{I}$ if (18) is solved.

- Global convergence and local superlinear convergence of this algorithm was analyzed in [14]. We usually observe the superlinear convergence in vicinity of the solution. In addition, it is possible to extend this algorithm for contact problems of elasto-plastic bodies, see [13, 14].

4 The modified SSR method in an algebraic form

To define the modified SSR method on the algebraic level, we arise from the system (7) and parametrize the function F using the factor λ introduced in Section 2:

$$\text{find} \quad \tilde{u}(\lambda) \in \mathbb{R}^n : \quad F_\lambda(\tilde{u}(\lambda)) = b, \tag{22}$$

Let us note that $F_\lambda = F$ if $\lambda = 1$. We shall also consider functions F^o_λ, $\mathcal{I}_\lambda$ and $\mathcal{I}_{\infty,\lambda}$ corresponding to the functions F^o, $\mathcal{I}$ and $\mathcal{I}_\infty$ introduced in Section 3. Then the minimization formulation of the problem (22) reads

$$\mathcal{J}_\lambda(\tilde{u}(\lambda)) \leq J_\lambda(v) \quad \forall v \in \mathbb{R}^n, \quad \mathcal{J}_\lambda(v) := \mathcal{I}_\lambda(v) - b^\top v. \tag{23}$$

The safety factor λ^* for the SSR method can be defined as follows:

$$\lambda^* \;=\; \text{maximum of} \;\; \lambda > 0 \;\; \text{such that the solution } \tilde{u}(\lambda) \text{ of (22) exists.} \tag{24}$$

To find λ^* one can straightforwardly increase λ by the following algorithm:

Algorithm SSRM

1. Initialization: Set, e.g., $\lambda_0 = 0.8$, $\delta\lambda_1 = 0.05$, $\delta\lambda_{min} = 0.001$, $k = 1$

2. While $\delta\lambda_k > \delta\lambda_{min}$:

   - Set $\lambda_k = \lambda_{k-1} + \delta\lambda_k$
   - Solve $F_{\lambda_k}(\tilde{u}(\lambda_k)) = b$ by the semismooth Newton method with damping, i.e.,

     $$u^{j+1} = u^j + \alpha_j s^j, \quad F^o_{\lambda_k}(u^j)s^j = b - F_{\lambda_k}(u^j), \quad \alpha_j = \arg\min_{\alpha\in[0,1]} \mathcal{J}_{\lambda_k}(u^j + \alpha s^j)$$

   - If the method does not converge, set $\delta\lambda_k := \delta\lambda_k/2$, $\lambda_k := \lambda_{k-1} + \delta\lambda_k$, and solve the system above again with updated λ_k. Otherwise, set $k := k + 1$.

One can see that the increments $\delta\lambda_k$ are reduced according the convergence of the semismooth Newton method with damping. Let us complete that the Newton-like method in Algorithm 4.1 is initiated by the solutions from previous steps and a fixed number of iteration is prescribed as usual. In addition, if the matrix $F^o_{\lambda_k}(u^j)$ is singular or ill-posed then we regularize it by $(1-\beta)F^o(u^j)+\beta K_{el}$, where $0<\beta<<1$ and K_{el} is the elastic stiffness matrix. If this Newton-like method convergences for some λ_k then one can expect that $\lambda_k \leq \lambda^*$. However, if the opposite is true, we cannot uniquely decide whether $\lambda_k > \lambda^*$ or not.

In Section 3, we introduced the limit analysis problem for the LL method using the cost function $\mathcal{I}_\infty$. In [5, 17], it was shown that a similar *kinematic limit problem for the SSR method* exists and reads as follows:

$$\lambda^* = \sup_{\lambda\geq 0}\left\{\lambda + G_\infty(\lambda)\right\}, \quad G_\infty(\lambda) := \inf_{v\in\mathbb{R}^n}\left[\mathcal{I}_{\infty,\lambda}(v) - b^\top v\right] \tag{25}$$

where

$$\mathcal{I}_{\infty,\lambda}(v) = \lim_{\omega\to+\infty}\mathcal{I}_{\omega,\lambda}(v), \quad \mathcal{I}_{\omega,\lambda}(v) = \frac{1}{\omega}\mathcal{I}_\lambda(\omega v), \quad \forall v\in\mathbb{R}^n. \tag{26}$$

Similarly as in Section 3, we derive the relationship between the definitions (25) and (24) of λ^* using the regularization parameter ω. For more details, we refer to [5].

To regularize the problem, it suffices to replace the function $\mathcal{I}_{\infty,\lambda}$ with $\mathcal{I}_{\omega,\lambda}$. Then, we arrive at the following approximation of (25) for given ω: *find* $\lambda(\omega) > 0$ *such that*

$$\lambda(\omega) + G_\omega(\lambda(\omega)) = \max_{\lambda\geq 0}\left\{\lambda + G_\omega(\lambda)\right\}, \quad G_\omega(\lambda) := \inf_{v\in\mathbb{R}^n}\left[\mathcal{I}_{\omega,\lambda}(v) - b^\top v\right]. \tag{27}$$

It holds that $\lambda(\omega)\leq\lambda^*$ and $\lim_{\omega\to+\infty}\lambda(\omega)=\lambda^*$, see [5]. Further, if $\lambda<\lambda^*$ then we have:

$$G_\omega(\lambda) = \frac{1}{\omega}\min_{v\in\mathbb{R}^n}\left[\mathcal{I}_\lambda(v) - b^\top v\right] = \frac{1}{\omega}\min_{v\in\mathbb{R}^n}\mathcal{J}_\lambda(v) = \frac{1}{\omega}\mathcal{J}_\lambda(\tilde{u}(\lambda)) =: \frac{1}{\omega}G_1(\lambda), \tag{28}$$

where $\tilde{u}(\lambda)$ solves (22) and also (23). Therefore, the value $G_\omega(\lambda)$ is inversely proportional to the parameter ω for any $\lambda<\lambda^*$.

The function $\omega\mapsto\lambda(\omega)$ has analogous properties as the function $\omega\mapsto t(\omega)$ sketched in Figure 1. We can construct values of this function, for example, as follows:

1. By Algorithm SSRM, construct the sequence $\lambda_0<\lambda_1<\ldots<\lambda_N$ and find the corresponding solutions $\tilde{u}(\lambda_i)$, $i=0,1,\ldots,N$, of the system (22).

2. Compute the values $G_1(\lambda_i)=\mathcal{J}_\lambda(\tilde{u}(\lambda_i))$, $i=0,1,\ldots,N$.

3. Find the values ω_i, $i=0,1,\ldots,N$, satisfying

$$\lambda_i + \frac{1}{\omega_i}G_1(\lambda_i) \geq \lambda_j + \frac{1}{\omega_i}G_1(\lambda_j) \quad \forall j=0,1,\ldots,N.$$

5 How to reduce numerical overestimation of FoS

In previous sections, we presented numerical methods for the determination of the safety factors t^* and λ^* for the modified LL and SSR methods. These safety factors can be overestimated, especially if lower-order finite elements are considered. The overestimation of safety factors is undesirable. In this section, we show how to suppress it by mesh refinements.

First, it is important to note that the definitions of t^*, $t(\omega)$, λ^*, and $\lambda(\omega)$ presented in Sections 3 and 4 can be extended to an infinite-dimensional functional space as was shown in [5, 8]. In order to study the dependence of t^*, $t(\omega)$, λ^*, and $\lambda(\omega)$ on the discretization parameter h, we use the notation t_h^*, $t_h(\omega)$, λ_h^*, and $\lambda_h(\omega)$ and let the original notation for the infinite-dimensional functional space. The convergence $\lim_{h\to 0} t_h(\omega) = t(\omega)$ hold for any $\omega > 0$ as was proven in [8]. In addition, $t(\omega)$ is a lower bound of t^*. However, the convergence $\lim_{h\to 0} t_h^* = t^*$ does not hold, in general, see Figure 2. It explains the observed overestimation which was documented, for example, in [18] where a posteriori numerical error analysis was used. Similar convergence results are expected also for the values λ_h^* and $\lambda_h(\omega)$ on basis of the theoretical results from [5].

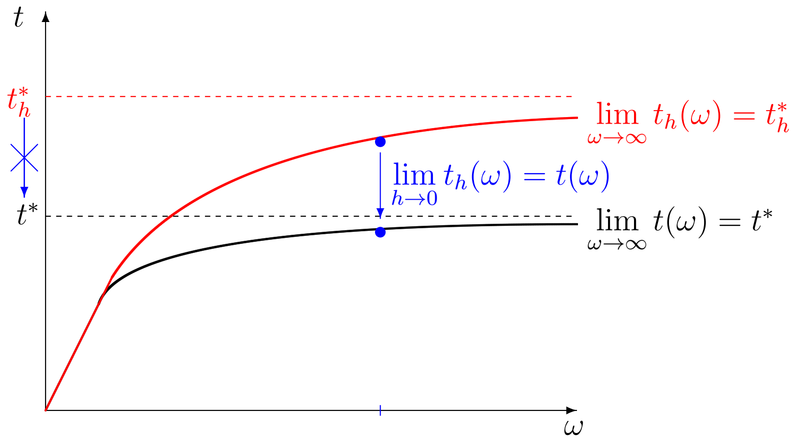

Figure 2: Convergence results with respect to the parameters h and ω.

From the curves in Figure 2, one can see that the convergence $\lim_{h\to 0} t_h(\omega) = t(\omega)$ is faster for smaller values of ω than for higher ones. Therefore, we suggest the following methodology:

1. Construct the curves $\omega \to t_h(\omega)$ and $\omega \to \lambda_h(\omega)$ for 2-3 finite element meshes with different densities.

2. Determine the maximal value of $\bar{\omega}$ for which the values $t_h(\bar{\omega})$ or $\lambda_h(\bar{\omega})$ are almost insensitive of the mesh density. These values are convenient for reliable estimation of t^* and λ^*, respectively.

Let us complete that this methodology was successfully tested in [8] for different perfectly plastic models and the LL method. This methodology has not been tested for the SSR method. This is a topic for an ongoing work. For mesh adaptive strategies, we refer to [2, 3, 5, 10].

In [9], there was introduced another method preventing the overestimation of t^*. The idea is to truncate unbounded failure surfaces (like the Mohr-Coulomb one) to be bounded. For bounded sets B (see Section 2), no significant overestimation of t^* occurs because the convergence $\lim_{h\to 0} t_h^* = t^*$ holds.

6 Conclusion

This contribution was focused on reliable computation of safety factors in slope stability and other applications. We presented the modified limit load and shear strength reduction methods. We proposed the methodology how to eliminate spurious numerical oscillations and the overestimation of the safety factors. The methodology was documented on numerical examples with the LL methods in previous author's work and will be tested soon for the SSR method.

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