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New Variant of the Semi-Monotonic Augmented Lagrangian Algorithm

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Abstract

SMALE is an efficient algorithm for solving quadratic programming problems with simple bounds and linear equality constraints. There are two variants of this method: one updates the parameter for precision control of an inner solver by a factor less than one (the preferable variant, as it does not change the Hessian via penalty update), and the other updates the penalty by a factor greater than one (resulting in a lower number of outer iterations and fewer Hessian multiplications in the inner solver). We use the MPRGP algorithm as an inner solver for solving bound-constrained quadratic programming problems.

We introduce a new theoretically supported variant that updates both these parameters: multiplying the penalty by a factor greater than one and multiplying the parameter for precision control for the MPRGP stopping criterion by the square root of this factor. The larger penalty accelerates the outer loop, while the larger parame-

ter for precision control accelerates the inner solver. Numerical experiments with the Total-FETI method demonstrate the effectiveness of this new variant.

Keywords: quadratic programming, augmented Lagrangian, Lagrange multipliers, FETI method, contact problem, equality constraint, SMALE.

1 Introduction

SMALE-M and SMALE-ρ (Semi-Monotonic Augmented Lagrangian method for Equality constraints) algorithms are efficient tools for solving the quadratic programming (QP) problems with equality constraint and simple bounds [1]. These algorithms consist of an outer loop for the update of parameters M or ρ, an update of the Lagrange multipliers for equality constraint and MPRGP (Modified Proportioning with Reduced Gradient Projection) algorithm [3] used as an inner solver for bound constrained QP problems with penalized equality constraint. These algorithms are implemented into our scalable in-house PERMON library [4] based on PETSc [6]. SMALE terminates if the norm of the equality constraint violation and the norm of the projected gradient (see below) are sufficiently small compared to the norm of the right-hand side multiplied by the relative tolerance. MPRGP terminates if the norm of the projected gradient is less than the norm of the violation of the equality constraint multiplied by the M parameter. Parameter M is fixed for SMALE-ρ while penalty ρ increases depending on the augmented Lagrangian growth. In SMALE-M depending on the augmented Lagrangian growth, M decreases while ρ is fixed. The larger penalty ρ accelerates the outer loop, while the larger parameter M accelerates the inner solver. This paper deals with the new theoretically supported SMALE-ρM variant increasing both M and ρ parameters, which reduces both the number of outer and inner iterations. The efficiency of the new method is demonstrated by numerical experiments with a model contact problem solved using the TFETI (Total Finite Element Tearing and Interconnect) method [5].

2 The MPRGP Algorithm

MPRGP represents an efficient algorithm for the solution of convex QP problems with box constraints, i.e. for

$$\arg\min_{\boldsymbol{x}} \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} \quad \text{s.t.} \quad \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, $\boldsymbol{x}$ is the solution, $\boldsymbol{b}$ is the right-hand side, $\boldsymbol{l}$ and $\boldsymbol{u}$ is the lower and upper bound, respectively. The basic version belonging to active set-based methods can be considered a modification of the Polyak algorithm [1]. MPRGP performs three types of steps - the classical conjugate gradient (CG) step, the partial CG step to the bound followed by an expansion step (expanding

the active set) and the proportioning step (reducing the active set). Modifications of the active set are done using the components of the gradient.

Let $\boldsymbol{g} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$ be the gradient. Then we can define component-wise (for $j \in \{1, 2, \ldots, n\}$) gradient splitting that is computed after each gradient evaluation. The free gradient is defined as

$$\boldsymbol{g}_j^f = \begin{cases} 0 & \text{if} \quad \boldsymbol{x}_j = \boldsymbol{l}_j \quad \text{or} \quad \boldsymbol{x}_j = \boldsymbol{u}_j, \\ \boldsymbol{g}_j & \text{otherwise.} \end{cases}$$

The reduced free gradient is

$$\boldsymbol{g}_j^r = \begin{cases} 0 & \text{if} \quad \boldsymbol{x}_j = \boldsymbol{l}_j \quad \text{or} \quad \boldsymbol{x}_j = \boldsymbol{u}_j, \\ \min\left(\frac{\boldsymbol{x}_j - \boldsymbol{l}_j}{\overline{\alpha}}, \boldsymbol{g}_j\right) & \text{if} \quad \boldsymbol{l}_j < \boldsymbol{x}_j < \boldsymbol{u}_j \quad \text{and} \quad \boldsymbol{g}_j > 0, \\ \max\left(\frac{\boldsymbol{x}_j - \boldsymbol{u}_j}{\overline{\alpha}}, \boldsymbol{g}_j\right) & \text{if} \quad \boldsymbol{l}_j < \boldsymbol{x}_j < \boldsymbol{u}_j \quad \text{and} \quad \boldsymbol{g}_j \leq 0, \end{cases}$$

where $\overline{\alpha} \in (0, 2||\boldsymbol{A}||^{-1}]$ is used as a step length in the expansion step. The definition of the chopped gradient is

$$\boldsymbol{g}_j^c = \begin{cases} 0 & \text{if} \quad \boldsymbol{l}_j < \boldsymbol{x}_j < \boldsymbol{u}_j, \\ \min(\boldsymbol{g}_j, 0) & \text{if} \quad \boldsymbol{x}_j = \boldsymbol{l}_j, \\ \max(\boldsymbol{g}_j, 0) & \text{if} \quad \boldsymbol{x}_j = \boldsymbol{u}_j. \end{cases}$$

Finally, the projected gradient is defined as $\boldsymbol{g}^P = \boldsymbol{g}^f + \boldsymbol{g}^c$. Its norm decrease is the natural stopping criterion of the algorithm.

Let the projection onto the feasible set $\Omega = \{\boldsymbol{x} : \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$ be defined as

$$[P_\Omega(\boldsymbol{x})]_j = \min(\boldsymbol{u}_j, \max(\boldsymbol{l}_j, \boldsymbol{x}_j)).$$

Now we have all the necessary ingredients to summarise MPRGP in Algorithm 1. The algorithm has been proven to enjoy an R-linear rate of convergence given by the bound on the spectrum of the Hessian matrix [1].

3 The SMALE Algorithms

The SMALE algorithm eliminates the requirement to set the proper penalty value ρ and extends an inner solver by an outer loop updating the Lagrange multiplier for the equality constraint. The outer loop (potentially) updates the penalty parameter, the inner solver stopping criterion and the right-hand side. Then the appropriate inner solver is called to solve QP with an eliminated equality constraint.

Our goal is to solve QP with box and linear equality constraints

$$\arg\min_{\boldsymbol{x}} \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} \quad \text{s.t.} \quad \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \quad \text{s.t.} \quad \boldsymbol{B_E}\boldsymbol{x} = \boldsymbol{o}, \tag{2}$$

Algorithm 1: MPRGP

Input: $\boldsymbol{A}$, $\boldsymbol{x}^0 \in \Omega$, $\boldsymbol{b}$, $\Gamma > 0$, $\overline{\alpha} \in (0, 2||\boldsymbol{A}||^{-1}]$

$\boldsymbol{g} = \boldsymbol{A}\boldsymbol{x}_0 - \boldsymbol{b}$, $\boldsymbol{p} = \boldsymbol{g}^f(\boldsymbol{x}^0)$, $k = 0$

while $||\boldsymbol{g}^P(\boldsymbol{x}^k)||$ *is not small***:**

  if $||\boldsymbol{g}^c(\boldsymbol{x}^k)||^2 \leq \Gamma^2 \boldsymbol{g}^r(\boldsymbol{x}^k)^T \boldsymbol{g}^f(\boldsymbol{x}^k)$**:**

    $\alpha_f = \max\{\alpha_{cg} : \boldsymbol{x}^k - \alpha_{cg}\boldsymbol{p}\}$

    $\alpha_{cg} = \boldsymbol{g}^T\boldsymbol{p}/\boldsymbol{p}^T\boldsymbol{A}\boldsymbol{p}$

    if $\alpha_{cg} < \alpha_f$**:**

      // CG step

      $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \alpha_{cg}\boldsymbol{p}$

      $\boldsymbol{g} = \boldsymbol{g} - \alpha_{cg}\boldsymbol{A}\boldsymbol{p}$; $\beta_{cg} = \boldsymbol{g}^f(\boldsymbol{x}^{k+1})^T\boldsymbol{A}\boldsymbol{p}/\boldsymbol{p}^T\boldsymbol{A}\boldsymbol{p}$

      $\boldsymbol{p} = \boldsymbol{g}^f(\boldsymbol{x}^{k+1}) - \beta_{cg}\boldsymbol{p}$

    else:

      // Expansion step

      $\boldsymbol{x}^{k+\frac{1}{2}} = \boldsymbol{x}^{k+1} - \alpha_f\boldsymbol{p}$

      $\boldsymbol{g} = \boldsymbol{g} - \alpha_f\boldsymbol{p}$

      $\boldsymbol{x}^{k+1} = \boldsymbol{x}^{k+\frac{1}{2}} - \overline{\alpha}\boldsymbol{g}^r(\boldsymbol{x}^{k+\frac{1}{2}})$

      $\boldsymbol{g} = \boldsymbol{A}\boldsymbol{x}^{k+1} - \boldsymbol{b}$

      $\boldsymbol{p} = \boldsymbol{g}^f(\boldsymbol{x}^{k+1})$

  else:

    // Proportioning step

    $\alpha_{cg} = \boldsymbol{g}^T\boldsymbol{g}^c(\boldsymbol{x}^k)/\boldsymbol{g}^c(\boldsymbol{x}^k)^T\boldsymbol{A}\boldsymbol{g}^c(\boldsymbol{x}^k)$

    $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \alpha_{cg}\boldsymbol{g}^c(\boldsymbol{x}^k)$

    $\boldsymbol{g} = \boldsymbol{g} - \alpha_{cg}\boldsymbol{A}\boldsymbol{g}^c(\boldsymbol{x}^k)$

    $\boldsymbol{p} = \boldsymbol{g}^f(\boldsymbol{x}^{k+1})$

  $k = k + 1$

Output: $\boldsymbol{x}^k$

where $\boldsymbol{B_E} \in \mathbb{R}^{m\times n}$. The right-hand side of the linear equality constraint is assumed to be zero, but linear equality-constrained QP with a nonzero right-hand side can always be homogenized.

The MPRGP algorithm is used as the inner solver for the box-constrained subproblem. Inner solvers iterates while

$$||\boldsymbol{g}^P|| \geq \min(M_k||\boldsymbol{B_E}\boldsymbol{x}||, \eta).$$

Traditionally, enforcing the equality constraint can be achieved by the quadratic penalty $\rho\boldsymbol{B_E}^T\boldsymbol{B_E}$ being part of the Hessian

$$\boldsymbol{A} + \rho\boldsymbol{B_E}^T\boldsymbol{B_E}.$$

The convergence is then very sensitive to spectral properties of $\rho\boldsymbol{B_E}^T\boldsymbol{B_E}$ and setting the right value of the penalty ensuring sufficient fulfilment of this equality constraint,

and not spoiling significantly the conditioning of this Hessian. A large penalty guarantees a more accurate fulfilment of the equality constraint but destroys the convergence rate. The spectral properties can be improved by multiplying the equality constraint by a transformation matrix $\boldsymbol{T}$ defining the orthonormalization of rows of $\boldsymbol{B_E}$, so that

$$(\boldsymbol{T}\boldsymbol{B_E})^T\boldsymbol{T}\boldsymbol{B_E} = \boldsymbol{B_E}^T(\boldsymbol{B_E}\boldsymbol{B_E}^T)^{-1}\boldsymbol{B_E} = \boldsymbol{Q_E}$$

is the projector onto $Im\boldsymbol{B_E}^T$ and the Hessian is then given as

$$\boldsymbol{A} + \rho\boldsymbol{Q_E}.$$

Another significant improvement can be achieved via enforcing the equality constraint by the orthogonal projector

$$\boldsymbol{P_E} = \boldsymbol{I} - \boldsymbol{Q_E}$$

onto $Ker\boldsymbol{B_E}$. However, due to projection P_Ω onto the feasible set in the MPRGP algorithm in its expansion step, the projection $\boldsymbol{P_E}$ is not sufficient to enforce the equality constraint and the Hessian still has to be equipped with the penalized term $\rho\boldsymbol{Q_E}$, i.e., our Hessian is

$$\boldsymbol{P_E}\boldsymbol{A}\boldsymbol{P_E} + \rho\boldsymbol{Q_E}.$$

3.1 SMALE-M and SMALE-ρ variants

SMALE algorithm has two basic variants, namely SMALE-M and SMALE-ρ. It is recommended in both variants to start with a small penalty ρ and increase ρ or reduce M by a factor β if an increase of the augmented Lagrangian

$$L(\boldsymbol{x}, \boldsymbol{\mu}, \rho) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} + \boldsymbol{\mu}^T\boldsymbol{B_E}\boldsymbol{x} + \frac{\rho}{2}||\boldsymbol{B_E}\boldsymbol{x}||^2$$

in an outer loop is not sufficient, see Algorithms 2 and 3. The recommended variant is SMALE-M, as it does not change the Hessian matrix and does not require recomputation of fixed step-length $\bar{\alpha}$ for the expansion step.

Algorithm 2: SMALE-M variant.

Initialize: $\boldsymbol{x}_0$, $\beta > 1$, $M_0 > 0$, $\rho_0 > 0$, $\boldsymbol{\mu}_0 = \boldsymbol{o}$, $k = 0$
while $||\boldsymbol{g}^P(\boldsymbol{x}_k, \boldsymbol{\mu}_k, \rho_k)|| > \epsilon||\boldsymbol{b}|| \vee ||\boldsymbol{B}\boldsymbol{x}_k|| > \epsilon||\boldsymbol{b}||$**:**
  $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k + \rho_k\boldsymbol{B}\boldsymbol{x}_k$
  find $\boldsymbol{x}_{k+1} \geq \boldsymbol{l}$ and $\boldsymbol{x}_{k+1} \leq \boldsymbol{u}$ such that
    $||\boldsymbol{g}^P(\boldsymbol{x}_{k+1}, \boldsymbol{\mu}_{k+1}, \rho_k)|| \leq \min(M_k||\boldsymbol{B}\boldsymbol{x}_{k+1}||, \eta)$
  if $L(\boldsymbol{x}_{k+1},\ \boldsymbol{\mu}_{k+1},\ \rho_k) \leq L(\boldsymbol{x}_k,\ \boldsymbol{\mu}_k,\ \rho_{k-1}) + \frac{1}{2}\rho_k||\boldsymbol{B}\boldsymbol{x}_{k+1}||^2$**:**
    $M_{k+1} = M_k/\beta$
  $k = k + 1$

Algorithm 3: SMALE-ρ variant.

Initialize: $\boldsymbol{x}_0$, $\beta > 1$, $M_0 > 0$, $\rho_0 > 0$, $\boldsymbol{\mu}_0 = \boldsymbol{o}$, $k = 0$

while $||\boldsymbol{g}^P(\boldsymbol{x}_k, \boldsymbol{\mu}_k, \rho_k)|| > \epsilon||\boldsymbol{b}|| \vee ||\boldsymbol{B}\boldsymbol{x}_k|| > \epsilon||\boldsymbol{b}||$**:**

  $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k + \rho_k \boldsymbol{B}\boldsymbol{x}_k$

  find $\boldsymbol{x}_{k+1} \geq \boldsymbol{l}$ and $\boldsymbol{x}_{k+1} \leq \boldsymbol{u}$ such that
  $||\boldsymbol{g}^P(\boldsymbol{x}_{k+1}, \boldsymbol{\mu}_{k+1}, \rho_k)|| \leq \min(M_k||\boldsymbol{B}\boldsymbol{x}_{k+1}||, \eta)$

  if $L(\boldsymbol{x}_{k+1},\ \boldsymbol{\mu}_{k+1},\ \rho_k) \leq L(\boldsymbol{x}_k,\ \boldsymbol{\mu}_k,\ \rho_{k-1}) + \frac{1}{2}\rho_k||\boldsymbol{B}\boldsymbol{x}_{k+1}||^2$**:**

    $\rho_{k+1} = \beta\rho_k$

  $k = k + 1$

3.2 SMALE-ρM

Theorem 3.1 in [2] says that there exist bound on M_k generated by SMALE-M algorithm

$$M_k \geq \min(M_0, \sqrt{\rho\lambda_{min}}/\beta)$$

with λ_{min} denoting the smallest eigenvalue of matrix A.

Considering this theorem, there exists index k such that

$$M_k/\sqrt{\rho_k} \geq \sqrt{\lambda_{min}}/\beta = const.$$

Update of ρ_k by factor β allows us update of M_k by factor $\sqrt{\beta}$ to keep their ratio constant, i.e. M_k and ρ_k remain balanced also after multiplication by $\sqrt{\beta}$ and β, respectively. This is an idea behind new SMALE-ρM variant, see Alg. 4, when larger penalty ρ accelerates an outer loop and affects larger augmented Lagrangian increase and larger M parameter accelerates an inner loop via its stopping criterium requiring lower precision on the projected gradient $\boldsymbol{g}^P$.

Algorithm 4: Improved SMALE-ρM variant.

Initialize: $\boldsymbol{x}_0$, $\beta > 1$, $M_0 > 0$, $\rho_0 > 0$, $\boldsymbol{\mu}_0 = \boldsymbol{o}$, $k = 0$

while $||\boldsymbol{g}^P(\boldsymbol{x}_k, \boldsymbol{\mu}_k, \rho_k)|| > \epsilon||\boldsymbol{b}|| \vee ||\boldsymbol{B}\boldsymbol{x}_k|| > \epsilon||\boldsymbol{b}||$**:**

  $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k + \rho_k \boldsymbol{B}\boldsymbol{x}_k$

  find $\boldsymbol{x}_{k+1} \geq \boldsymbol{l}$ and $\boldsymbol{x}_{k+1} \leq \boldsymbol{u}$ such that
  $||\boldsymbol{g}^P(\boldsymbol{x}_{k+1}, \boldsymbol{\mu}_{k+1}, \rho_k)|| \leq \min(M_k||\boldsymbol{B}\boldsymbol{x}_{k+1}||, \eta)$

  if $L(\boldsymbol{x}_{k+1},\ \boldsymbol{\mu}_{k+1},\ \rho_k) \leq L(\boldsymbol{x}_k,\ \boldsymbol{\mu}_k,\ \rho_{k-1}) + \frac{1}{2}\rho_k||\boldsymbol{B}\boldsymbol{x}_{k+1}||^2$**:**

    $\rho_{k+1} = \beta\rho_k$ and $M_{k+1} = \sqrt{\beta}M_k$

  $k = k + 1$

4 Numerical experiments with TFETI for contact problems

Let us consider the spatial domain Ω which is decomposed into non-overlapping subdomains. Then virtually arbitrary Finite Element Method (FEM) implementation can be used to generate the subdomain stiffness matrices $\boldsymbol{K}^s$ and the subdomain load vectors $\boldsymbol{f}^s$ as sequential data for each subdomain Ω^s, $s = 1, \ldots, N_S$ independently.

The original primal problem is

$$\arg\min_{\boldsymbol{u}} \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f}^T\boldsymbol{u} \text{ s.t. } \boldsymbol{B}_I\boldsymbol{u} \leq \boldsymbol{o} \text{ and } \boldsymbol{B}_E\boldsymbol{u} = \boldsymbol{o}, \tag{3}$$

where $\boldsymbol{K} = diag(\boldsymbol{K}_1, \ldots, \boldsymbol{K}_{N_S})$ is global stiffness matrix, $\boldsymbol{f} = \left[\boldsymbol{f}_1^T, \ldots, \boldsymbol{f}_{N_S}^T\right]^T$ is global right hand side, $\boldsymbol{u}$ is unknown displacement, $\boldsymbol{B}_I$ represents non-penetration condition, and $\boldsymbol{B}_E$ glues the subdomains together. The primal problem is transformed into dual one

$$\arg\min_{\boldsymbol{\lambda}} \frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{P}\boldsymbol{F}\boldsymbol{P}\boldsymbol{\lambda} - \boldsymbol{\lambda}^T\boldsymbol{P}\boldsymbol{d} \text{ s.t. } \boldsymbol{\lambda}_I \geq -\tilde{\boldsymbol{\lambda}}_I \text{ and } \boldsymbol{G}\boldsymbol{\lambda} = \boldsymbol{o}, \tag{4}$$

where

$$\boldsymbol{F} = \boldsymbol{B}\boldsymbol{K}^\dagger\boldsymbol{B}^T,\ \boldsymbol{G} = \boldsymbol{R}^T\boldsymbol{B}^T,\ \boldsymbol{P} = \boldsymbol{I} - \boldsymbol{G}^T(\boldsymbol{G}\boldsymbol{G}^T)^{-1}\boldsymbol{G},$$

$$\boldsymbol{e} = \boldsymbol{R}^T\boldsymbol{f},\ \tilde{\boldsymbol{\lambda}} = \boldsymbol{G}^T(\boldsymbol{G}\boldsymbol{G}^T)^{-1}\boldsymbol{e},\ \boldsymbol{d} = \boldsymbol{B}\boldsymbol{K}^\dagger\boldsymbol{f} - \boldsymbol{F}\tilde{\boldsymbol{\lambda}},$$

$\boldsymbol{K}^\dagger$ denotes a left generalized inverse of $\boldsymbol{K}$, i.e. a matrix satisfying $\boldsymbol{K}\boldsymbol{K}^\dagger\boldsymbol{K} = \boldsymbol{K}$ and $\boldsymbol{R}$ is the null space matrix. The constraint matrix $\boldsymbol{B} = \left[\begin{smallmatrix}\boldsymbol{B}_I^T & \boldsymbol{B}_E^T\end{smallmatrix}\right]^T$ can be constructed so that it has a full rank, and then the Hessian $\boldsymbol{P}\boldsymbol{F}\boldsymbol{P}$ is positive definite with a relatively favourably distributed spectrum for the application of the CG method.

The chosen benchmark is two membranes semicoercive contact problem described in Figure 1. We use TFETI domain decomposition to solve this problem. Let us consider only one regular decomposition into 32 subdomains (4 in each direction per one membrane) with 81 elements per subdomain (9 in each direction) making a total of 3 200 degrees of freedom. The stopping tolerance of the outer solver (SMALE) is set to 10^{-8} relative to the right-hand side. Other initial parameters for SMALE are $M = ||\boldsymbol{A}||$, $\eta = 1.1||\boldsymbol{A}||$, $\rho = ||\boldsymbol{A}||$ and, parameter β for M or ρ update takes values from $\{2, 10\}$. We used standard expansion with fixed-length $\bar{\alpha} = 1/||\boldsymbol{A}||$. The results are presented in Table 1.

SMALE-ρM can significantly decrease the number of outer iterations (e.g. from 49 and 52 in SMALE-M and from 20 and 12 in SMALE-ρ to 12 and 8 in SMALE-ρM case) and the total number of Hessian multiplications (e.g. from 87 and 156 in SMALE-M and from 86 and 99 in SMALE-ρ to 75 and 86 in SMALE-ρM case).

Semicoercive QP problem

$$-\triangle u = f \quad \text{in} \quad \Omega = \Omega^1 \cup \Omega^2$$

$$u^i = 0 \quad \text{on} \quad \Gamma_u^i, \ i = 1, 2$$

$$\frac{\partial u^i}{\partial n_i} = 0 \quad \text{on} \quad \Gamma_f^i, \ i = 1, 2$$

$$u^2 - u^1 \geq 0 \quad \text{on} \quad \Gamma_c = \Gamma_c^1 = \Gamma_c^2$$

$$\frac{\partial u^2}{\partial n_2} \geq 0 \quad \text{on} \quad \Gamma_c$$

$$\frac{\partial u^2}{\partial n_2}(u^2 - u^1) = 0 \quad \text{on} \quad \Gamma_c$$

$$\frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} = 0 \quad \text{on} \quad \Gamma_c$$

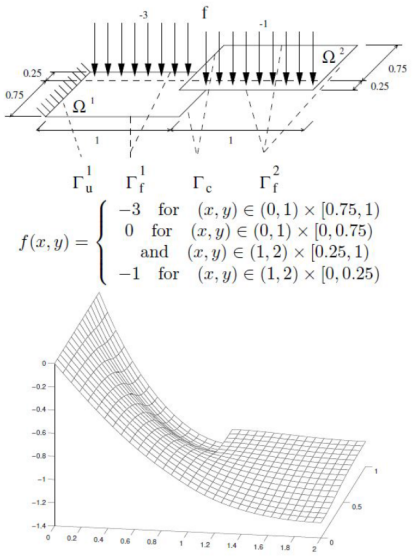

Figure 1: Model contact problem.

β	SMALE variant	#outer iters	#Hess. mult.	#CG st.	#Exp. st.	#Pro. st.
2	-M	49	87	81	3	0
	-ρ	20	86	80	3	0
	-ρM	12	75	69	3	0
10	-M	52	156	150	3	0
	-ρ	12	99	93	3	0
	-ρM	8	86	80	3	0

Table 1: Comparison of SMALE variants by the numbers of outer iterations, overall Hessian multiplications, CG, expansion, and proportioning steps for $\beta \in \{2, 10\}$.

5 Concluding remarks

The new SMALE-ρM algorithm updating both parameters ρ and M keeping constant ratio $M_k/\sqrt{\rho_k}$ was investigated. The numerical results show that this version can outperform both standard versions SMALE-M and SMALE-ρ reducing the number of outer iterations by a factor of up to 6.5 and 1.7, respectively, and reducing the number of Hessian multiplications by a factor up to 1.8 and 1.2, respectively.

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