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Stochastic Projection Based Gradient Free PINN for Reliability Analysis of System using PDEM

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Abstract

In this paper, a reliability analysis of stochastic systems is presented using probability density evolution method (PDEM). In PDEM, generalized density evolution equations (GDEEs) are completely decoupled between physical and probability space, which is developed based on the idea of probability conservation. Using the GF-discrepancy technique, a collection of representative points of random variables are constructed in order to provide an accurate estimate of the probability density function. Sufficient precision requires a large number of sample points, which becomes computationally costly. Physics-informed neural network (PINN)-based PDEM is one of promising methods which reduce the computational cost. Beside the advantages of PINN for solving GDEEs in PDEM, PINN may suffer from gradient estimation using Automatic Differentiation. In this study, stochastic projection based PINN, a gradient free method, is a coupled framework of stochastic projection theory and traditional PINN, for solving GDEEs. To illustrate the efficiency of the method, two numerical examples are investigated for estimating probability density function which is utilized for reliability analysis of stochastic systems.

Keywords: reliability analysis, probability density evolution method, physics-informed neural network, stochastic projection theory, partial differential equation, uncertainty propagation.

1 Introduction

Reliability analysis of engineering structures is a crucial tool for calculating the probability of structural system failure, considering the uncertainties in the system and external forces. Various reliability methods have been proposed for determining failure probability, categorized into four sub-groups: analytical approximation methods, numerical sampling-based methods, surrogate-based methods, and numerical integration methods. The analytical approximation methods include the first-order and second-order reliability methods (FORM and SORM). These methods approximate the performance function around the most probable point (MPP) using first and second-order Taylor expansions [2]. However, they can yield erroneous results when there are multiple MPPs or when the performance function is nonlinear.

Numerical sampling-based methods, such as Monte Carlo simulation, subset simulation, are more robust compared to FORM and SORM. However, these methods are computationally expensive for high-fidelity models or when estimating low failure probabilities, as they require numerous model evaluations. To reduce computational costs, surrogate-based methods have gained popularity in recent decades. These methods use surrogate models to replace the original performance function with a few observations. Popular surrogate models include Kriging [3], polynomial chaos expansion [4], and artificial neural networks [5].

The numerical integration method estimates failure probability by integrating the probability density function (PDF) of the performance function over the failure domain. The PDF is estimated using statistical moments, including central moments [6], fractional moments [7], the point estimation method [8] etc. The probability density evolution method (PDEM) is a numerical integration technique proposed by Li and Chen [9]. PDEM is based on the principle of probability conservation. To estimate the probability density function (PDF) of the performance function using PDEM, an effective strategy is necessary to generate representative points for the random variables in the system. This strategy ensures that the representative points adequately cover the entire probability space, allowing PDEM to accurately estimate the PDF within the distribution domain. Various sampling strategies can be used, including the number-theoretical method, the tangent sphere method, the quasi-symmetrical point method, the GF discrepancy-based method, and the partially stratified sampling-based method. Among these, the GF discrepancy-based method [9] is the most common and popular for estimating the PDF using PDEM. While PDEM provides accurate PDF estimations, it requires a large number of representative points, which can be computationally expensive for high-fidelity models.

To reduce the computational burden, different machine learning algorithms have been proposed by different researchers. Among them, physics-informed neural network (PINN) have gained popularity in the recent decade. PINN, an emerging deep learning method proposed by Raissi and Karniadakis [11], incorporates physical conditions into neural networks. Similar to a standard feed-forward neural network, the loss function in PINN is formulated by incorporating the governing physical laws, represented as an ordinary differential equation (ODE) or a partial differential equation (PDE), along with the system's initial and boundary conditions. Recently, Das and Tesfamariam [12] proposed a couled framework between PINN and PDEM for reliability analysis of stochastic system. Although PINN models have shown promise as solutions for a wide range of physical phenomena, achieving high model accuracy and training efficiency often remains challenging. Also PINN employs a fully connected network that enforces physical constraints via residual loss calculated at collocation points. As the training loss is computed using Automatic Differentiation, the network must be differentiable to meet the requirements of the governing differential equations. To alleviate these issues, in this study, a stochastic projection based PINN is utilized for solving PDEM which combines the stochastic projection theory with traditional PINN and eliminates the computation of gradients in PINN.

The outline of the study is structured as follows. Section 2 describes the complete overview of the reliability analysis of the stochastic system, in which a brief description of PDEM and how PDEM is used to estimate the reliability of a system. Section 3 provides a brief description of stochastic projection based PINN. The numerical applications of the proposed method are presented in Section 4. Finally, Section 5 presents the concluding remarks.

2 Probability Density Evolution Method for Reliability Analysis

Consider a structural system with *N* degrees of freedom subjected to a dynamic external force, $\eta(t)$. The governing equation of motion is expressed as

$$\mathbf{M}(\mathbf{\Theta})\ddot{\mathbf{X}} + \mathbf{C}(\mathbf{\Theta})\dot{\mathbf{X}} + \mathbf{K}(\mathbf{\Theta})\mathbf{X} = \mathbf{\Gamma}\eta(\mathbf{\Theta}, t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ denote the mass, damping, and stiffness matrices of the structure, respectively. The structural responses i.e., displacement, velocity and acceleration are represented by $\mathbf{X}$, $\dot{\mathbf{X}}$, and $\ddot{\mathbf{X}}$, respectively. The random variables present in the system is denoted by $\mathbf{\Theta}$. The influence vector associated with the external excitation is represented by $\mathbf{\Gamma}$. Therefore, the state space formulation of Eq. 1 can be expressed as

$$\dot{\mathbf{Y}} = \mathbf{A}(\mathbf{Y}, \mathbf{\Theta}, t) \tag{2}$$

where the state vector $\mathbf{Y}$ is expressed as $\mathbf{Y} = \begin{bmatrix}\mathbf{X}^T & \dot{\mathbf{X}}^T\end{bmatrix}^T = \begin{bmatrix}Y_1 & \dots & Y_{2N}\end{bmatrix}^T$. The matrix $\mathbf{A}$ in Eq. 2 is written as

$$\mathbf{A} = \begin{bmatrix} \dot{\mathbf{X}}^T \\ [-\mathbf{M}^{-1}\mathbf{C}(\mathbf{\Theta})\dot{\mathbf{X}} - \mathbf{M}^{-1}\mathbf{K}(\mathbf{\Theta})\mathbf{X} + \mathbf{\Gamma}\mathbf{M}^{-1}\eta(\mathbf{\Theta}, t)]^T \end{bmatrix} \tag{3}$$

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2.1 Probability Density Evolution Method

The present investigation employs the probability density evolution technique (PDEM), as suggested by Li and Chen [9], to estimate the probability density function (PDF) of a stochastic process. The approach is formulated based on the principle of probability conservation. Therefore, the generalized density evolution equation (GDEE) in PDEM can be expressed as [9]

$$\frac{\partial p_{\mathbf{Y}\boldsymbol{\Theta}}(\mathbf{y},\boldsymbol{\theta},t)}{\partial t}+\sum_{i=1}^{2N}\dot{Y}_i(\boldsymbol{\theta},t)\frac{\partial p_{\mathbf{Y}\boldsymbol{\Theta}}(\mathbf{y},\boldsymbol{\theta},t)}{\partial y_i}=0 \tag{4}$$

where the joint PDF between $\mathbf{Y}$ and $\boldsymbol{\Theta}$ at any time instant t is denoted by $p_{\mathbf{Y}\boldsymbol{\Theta}}(\mathbf{y},\boldsymbol{\theta},t)$. The above equation can be expressed in reduced space when one physical quantity is considered,

$$\frac{\partial p_{Y\boldsymbol{\Theta}}(\mathbf{y},\boldsymbol{\theta},t)}{\partial t}+\dot{Y}(\boldsymbol{\theta},t)\frac{\partial p_{Y\boldsymbol{\Theta}}(\mathbf{y},\boldsymbol{\theta},t)}{\partial y}=0 \tag{5}$$

The initial condition of the above partial differential equation is expressed as

$$p_{Y\boldsymbol{\Theta}}(y,\boldsymbol{\theta},t)|_{t=0}=\delta\big[y-y(\boldsymbol{\theta},0)\big]p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \tag{6}$$

where $\delta(\cdot)$ represents the Dirac delta function. By solving Eq. 5 with an initial condition as in Eq. 6, the joint PDF Y and $\boldsymbol{\Theta}$ can be estimated which is used to calculate the PDF of Y, expressed as

$$p_Y(y,t)=\int_{\Omega_{\boldsymbol{\Theta}}}p_{Y\boldsymbol{\Theta}}(y,\boldsymbol{\theta},t)d\boldsymbol{\theta} \tag{7}$$

2.2 Estimation of First-passage Reliability using PDEM

The probability of failure (P_f) for a first-passage problem can be expressed as

$$P_f=\Pr\big\{|Y(t)|>Y_{\text{Thres}},\exists t\in[0,T]\big\} \tag{8}$$

where $\Pr(\cdot)$ represents the probability operator and threshold value of $Y(t)$ is denoted by Y_{Thres}. It is assumed that the process $Y(t)$ is evaluated within the time interval [0, T] where T is the maximum duration. Using the principle of equivalent extreme-value event [10], Eq. 8 can be reformulated as

$$P_f=\Pr\big\{Y_{\text{EEV}}\geq Y_{\text{Thres}}\big\}=\int_{Y_{\text{Thres}}}^{\infty}p_{Y_{\text{EEV}}}(y)dy \tag{9}$$

where the PDF of equivalent extreme-value (EEV) of $Y(t)$ is represented by $p_{Y_{\text{EEV}}}(y)$. Also, EEV of $Y(t)$ is the designer's choice. For instance, for maximum value of the process $Y(t)$, $Y_{\text{EEV}}=\max\limits_{t\in[0,T]}\big(|Y(t)|\big)$. As it is seen from Eq. 9 that $p_{Y_{\text{EEV}}}(y)$ needs to be estimated using Eq. 5. However, Y_{EEV} is independent of time, and Eq. 5 can not be used directly. Therefore, we assume a virtual stochastic process which is expressed as follows

$$\mathcal{V}(\tau) = Y_{\mathrm{EEV}} \sin(\omega\tau); \quad \tau \in [0, 1] \tag{10}$$

where $\omega = 5\pi/2$ [10]. It is seen from the above equation that when τ = 0, $\mathcal{V}(\tau) = 0$ and when τ = 1, $\mathcal{V}(\tau) = Y_{\mathrm{EEV}}$. Using Eq. 5, we can estimate the PDF of $\mathcal{V}(\tau)$ which is written as

$$\frac{\partial p_{\mathcal{V}}(\mathcal{V}, \tau)}{\partial \tau} + \dot{\mathcal{V}}(\tau)\frac{\partial p_{\mathcal{V}}(\mathcal{V}, \tau)}{\partial \mathcal{V}} = 0 \tag{11}$$

Once $p_{\mathcal{V}}(\mathcal{V}, \tau)$ is obtained, the PDF of Y_{EEV} is calculated as

$$p_{Y_{\mathrm{EEV}}}(y) = p_{\mathcal{V}}(\mathcal{V}, \tau)|_{\tau=1} \tag{12}$$

We are interested to solve Eq. 11 using physics-informed neural network, which is discussed in the following section.

3 Stochastic Projection Based Physics Informed Neural Network

A physics-informed neural network (PINN) is a type of feed-forward neural network that incorporates physical constraints into the network through its loss function [1]. The architecture of a PINN is similar to that of a standard feed-forward neural network, where the network's output is represented as a nested nonlinear transformation of the outputs obtained from the hidden layers.

$$l_0 = \mathbf{X}; \;\; l_i = \Phi(\mathbf{W}_i l_{i-1} + \mathbf{b}_i); \;\; y = l_n; \;\; \forall i \in [1, n] \tag{13}$$

where Φ denotes the activation function, a nonlinear operation. The network has n number of hidden layers and weights and bias of the network are denoted by $\mathbf{W}$ and $\mathbf{b}$, respectively. Using the concepts of a feed-forward neural network, a PINN is constructed where the weight and bias parameters are updated by minimizing a physics-informed loss function. In this study, we employ PINN to solve a nonlinear partial differential equation, which is expressed in the following form

$$\begin{aligned} &\frac{\partial p(y,t)}{\partial t} + \mathcal{N}[p(y,t); \chi] = 0; \quad y \in \Omega, t \in \mathcal{T} \\ &p(y,t) = \mathcal{G}(y,t); \;\; p(y,0) = \mathcal{H}(y) \quad y \in \Omega, t \in \mathcal{T} \end{aligned} \tag{14}$$

where $p(y,t)$ is the solution of Eq. 14 which is a function of a spatial variable, $y \in \Omega$, where Ω represents a space in $\mathbb{R}^D$ and a temporal variable, $t \in [0, T]$. The nonlinear differential operator of the above equation is denoted by $\mathcal{N}[p(y,t); \chi]$ in which χ is the coefficient of the differential operator. Also, $\mathcal{G}(y,t)$ and $\mathcal{H}(y)$ denote the boundary and initial conditions, respectively. The loss functions corresponding to Eq. 14 can be formulated in the following form:

$$\mathcal{L}_p(\lambda) = \frac{1}{N_p}\sum_{i=1}^{N_p}\left|\frac{\partial p(y_p^i, t_p^i;\lambda)}{\partial t} + \mathcal{N}[p(y_p^i, t_p^i);\chi]\right|^2$$

$$\mathcal{L}_0(\lambda) = \frac{1}{N_0}\sum_{i=1}^{N_0}\left|p(y_0^i, 0;\lambda) - \mathcal{H}(y_0^i, 0)\right|^2; \mathcal{L}_b(\lambda) = \frac{1}{N_b}\sum_{i=1}^{N_b}\left|p(y_b^i, t_b^i;\lambda) - \mathcal{G}(y_b^i, t_b^i)\right|^2 \tag{15}$$

where the residual of the differential equation, in Eq. 14 is denoted by $\mathcal{L}_p$. Also, the residual corresponding to initial and boundary conditions are represented by $\mathcal{L}_0$ and $\mathcal{L}_b$, respectively. In Eq. 15, λ is the vector consisting of the weight and bias parameters of the neural network. $\{y_p^i, t_p^i\}_{i=1}^{N_p}$ are the samples, drawn from the domain $\mathbf{x} \in \Omega, t \in \mathcal{T}$ where N_p denotes the total number of samples. Similarly, $\{y_0^i, \mathcal{H}_0^i = \mathcal{H}(y_0^i)\}_{i=1}^{N_0}$ and $\{y_b^i, t_b^i, \mathcal{G}_b^i = \mathcal{G}(y_b^i, t_b^i)\}_{i=1}^{N_b}$ are the samples corresponding to initial and boundary conditions, respectively where the samples sizes of the same are denoted by N_0 and N_b, respectively. Thus, a weighted sum of loss functions, written in Eq. 15, is constructed, which is given by

$$\mathcal{L}(\lambda) = \mathcal{L}_p(\lambda) + \beta_1\mathcal{L}_0(\lambda) + \beta_2\mathcal{L}_b(\lambda) \tag{16}$$

where β_1 and β_2 are the weight parameters to adjust the relative importance of each residual term. In this study, stochastic projection method is used to compute the gradient around the collocation points, as defined in Eq. 15. In this method, to compute the gradient at any point ($\bar{y}$, i.e., centre of the circle), we defined a neighbourhood of radius r_n, as shown in Fig. 1. Now within this neighbourhood, N number of samples, (y_i), are generated. Subsequently, gradient of network with respect to the input variable is expressed as

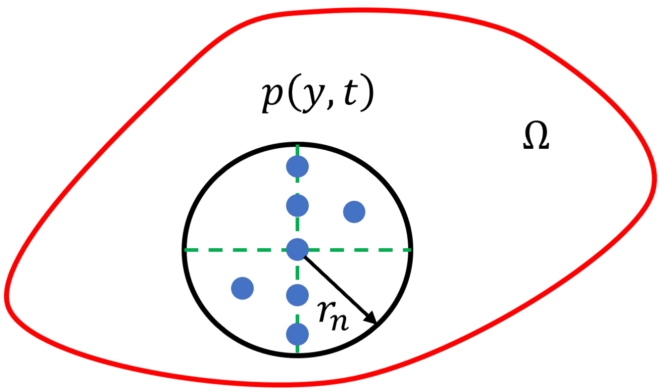


Figure 1: Diagram for stochastic projection method

$$\frac{\partial \mathcal{N}}{\partial y} = \frac{\frac{1}{N}\sum_i\{\mathcal{N}(y_i) - \mathcal{N}(\bar{y})\}(y_i - \bar{y})^T}{\frac{1}{N}\sum_i(y_i - \bar{y})(y_i - \bar{y})^T} \tag{17}$$

4 Numerical Results

Tho different numerical examples are presented to illustrate the reliability analysis using the probability density evolution method. Those are: (1) a four-branch problem in which the probability of failure is estimated from a series system; (2) reliability analysis of a shear building frame. To reduce the computational cost, physics-informed neural network is used, which approximates the original response surface of an equivalent extreme event. The gradient is computed based on stochastic projection theory.

4.1 Example 1:

As the first example, a four-branch problem is considered whose P_f is given by

$$\begin{aligned} P_f = \Pr\Big[-\frac{(\theta_1-\theta_2)^2}{10} + \frac{\theta_1+\theta_2}{\sqrt{2}} \geq 3 \,\cup -\frac{(\theta_1-\theta_2)^2}{10} - \frac{\theta_1+\theta_2}{\sqrt{2}} \geq 3 \\ \cup\, (\theta_1-\theta_2) \geq \frac{7}{\sqrt{2}} \,\cup\, (\theta_2-\theta_1) \geq \frac{7}{\sqrt{2}} \Big] \end{aligned} \tag{18}$$

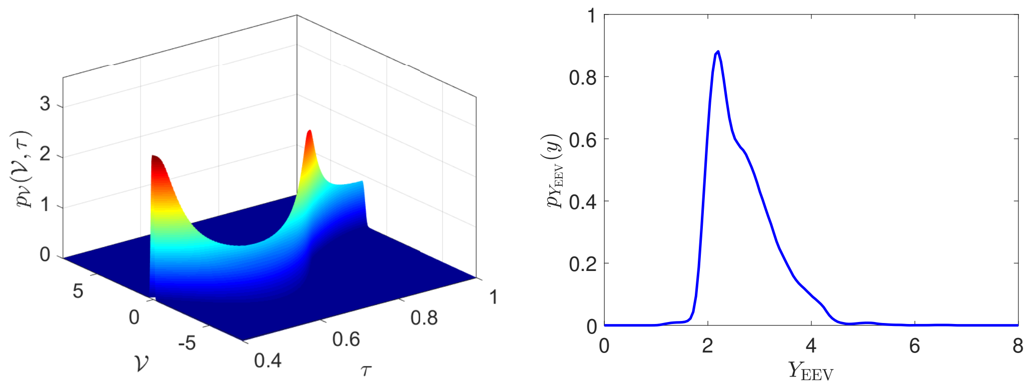
where θ_1 and θ_2 are the random variables which follow the standard Gaussian distribution. According to an equivalent extreme event, the above equation is equivalent to

$$P_f = \Pr\Big[Y_{\mathrm{EEV}} \geq 7/\sqrt{2}\Big] = \int_{7/\sqrt{2}}^{\infty} p_{Y_{\mathrm{EEV}}}(y)dy \tag{19}$$

where Y_{EEV} is given as

$$Y_{\mathrm{EEV}} = \max \left\{ \begin{matrix} -0.1(\theta_1-\theta_2)^2 + (\theta_1+\theta_2)/\sqrt{2} + 7/\sqrt{2} - 3 \\ -0.1(\theta_1-\theta_2)^2 - (\theta_1+\theta_2)/\sqrt{2} + 7/\sqrt{2} - 3 \\ (\theta_1-\theta_2) \\ (\theta_2-\theta_1) \end{matrix} \right\} \tag{20}$$

To train PINN, here a neural network with four hidden layers and 20 neurons in each hidden layer is considered. L-BFGS optimizer is used in this study in which a learning rate is taken as 0.1. To train the neural network, a randomly generated set of boundary and initial points along with equidistant collocation points are employed. The PDE loss at the collocation points, defined in Eq. 15 is computed using the stochastic projection method. As discussed in Section 3, the stochastic projection method leverages neighborhood information around a given point to evaluate the gradient. Therefore 2000 collocation points are generated to compute the gradient. Fig. 2(a) shows the PDF of $\mathcal{V}(\tau)$. The PDF of Y_{EEV} is obtained by setting τ to 1, as shown in Fig. 2(b). Once PDF is obtained, the probability of failure is estimated using Eq. 19. The failure probability is 2.48×10^{-3}. In comparison, a Monte Carlo simulation with 10^6 samples yields a failure probability of 2.43×10^{-3}, which is close to the value obtained using stochastic projection based PINN.



(a) (b)

Figure 2: PDF of (a) virtual stochastic process, $\mathcal{V}(\tau)$ and (b) equivalent extreme event, Y_{EEV}

4.2 Example 2:

We consider a 10-storey shear building frame for our next problem. The stiffness of each floor is assumed to follow a lognormal distribution with mean values of 1.962, 1.875, 1.758, 1.754, 1.662, 1.662, 1.662, 1.662, 1.662, and 1.662 ($\times 10^8$ N/m) from the bottom storey, with a coefficient of variation of 0.1. The lumped masses of each floor are : m_1 = 3.478, m_2 = 3.225, m_3 = 2.887, m_4 = 2.667, m_5 = 2.558, m_6 = 2.558, m_7 = 2.558, m_8 = 2.558, m_9 = 2.558, m_{10} = 2.558 ($\times 10^5$ kg), respectively, respectively. The height of each floor is 3.6 m. The structure is subjected to a ground motion, which is given as

$$\ddot{x}_g(t) = \beta_1 \ddot{x}_{NS}(t) + \beta_2 \ddot{x}_{EW}(t) \tag{21}$$

where $\ddot{x}_{NS}$ and $\ddot{x}_{EW}$ are the El Centro ground motion in N-S and E-W components, respectively. The coefficients β_1 and β_2 are normally distributed, whose mean and standard deviation are 2 and 0.2, respectively. The damping ratio is assumed to be 0.05. Hence, the problem has 12 random variables, i.e., $\theta = [k_1, k_2, \ldots, k_{10}, \beta_1, \beta_2]$. The probability of failure is given by

$$P_f = \Pr\left\{ \bigcup_{i=1}^{10} |Y_i(\theta,t)| \geq Y_{\text{Thres}}, \exists t \in [0,T] \right\} \tag{22}$$

where $Y_i(\theta, t)$ denotes interstory drift of the i-th story. The allowable interstory drift is denoted by Y_{Thres}, and the duration of the ground motion is denoted by T. According to the equivalent extreme event, Eq. 22 is equivalent to

$$P_f = \Pr\left[Y_{\text{EEV}} \geq Y_{\text{Thres}}\right] = \int_{Y_{\text{Thres}}}^{\infty} p_{Y_{\text{EEV}}}(y)dy \tag{23}$$

where Y_{EEV} is given as

$$Y_{\text{EEV}} = \max_{1 \leq i \leq 10} \left\{ \max_{t \in [0,T]} \left|Y_i(\theta,t)\right| \right\} \tag{24}$$

Like previously, to train PINN, here a neural network with six hidden layers and 30 neurons in each hidden layer is considered. L-BFGS optimizer is used in this study in which a learning rate is taken as 0.1. To train the neural network, a randomly generated set of boundary and initial points along with equidistant collocation points are employed. The PDE loss at the collocation points, defined in Eq. 15 is computed using the stochastic projection method. As discussed in Section 3, the stochastic projection method leverages neighborhood information around a given point to evaluate the gradient. Therefore 2500 collocation points are generated to compute the gradient. Fig. 3 shows PDF of extreme event as defined in Eq. 24.

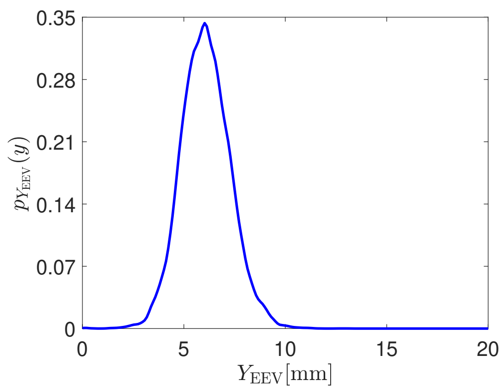


Figure 3: PDF of equivalent-extreme event, Y_{EEV}

5 Conclusions

The numerical investigations conducted in this study concentrate on reliability analysis of stochastic systems utilizing the probability density evolution method (PDEM). PDEM is computationally demanding for high-fidelity models due to its requirement for a large number of sample points. To alleviate this computational burden, physics-informed neural network (PINN) is employed. The key findings from this study are summarized as follows:

- PDEM efficiently estimates the joint probability density function of the equivalent-extreme event for a Multi-Degree-of-Freedom (MDOF) system by solving Generalized Differential Evolution Equations (GDEEs). This approach contrasts with traditional methods like the Fokker-Planck equation and Liouville equation, offering improved computational efficiency.

- The proposed stochastic projection based PINN overcomes the challenges during the computation of gradient in traditional PINN. This method produces the probability of failure close to Monte Carlo simulation.

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