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The Accuracy and Reliability of the Finite Element Method in Free Vibration Analysis of Beams and Frameworks

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Abstract

The accuracy and reliability of the finite element method (FEM) in free vibration analysis of beams and frameworks are assessed by the dynamic stiffness method (DSM). When applying FEM, individual beams and frameworks are modelled by using a progressively increasing number of elements. The accuracy and convergence of results are then judged against exact results using DSM. The limitation of FEM in which free vibration analysis cannot be carried out beyond the order of the mass and stiffness matrices, and the loss of accuracy in computing higher natural frequencies, are highlighted. The investigation has shown that in the low and medium frequency range, FEM is reliable, but in the high frequency range, FEM can become inaccurate and unreliable. Evaluation of modal analysis using FEM in the high frequency range is important, particularly for the statistical energy analysis method, because the modal density of structures in the high frequency range is usually high. Furthermore, the need for accurate and reliable high frequency vibration analysis for beams and frameworks is important in energy flow analysis. It is in this context that the assessment of FEM in free vibration analysis of beams and frameworks is expected to be most effective and useful.

Keywords: beam, framework, finite element method, dynamic stiffness method, Wittrick-Williams algorithm, shape function.

1 Introduction

The finite element method (FEM) is a well-recognised universal tool in structural analysis and design, which can solve multifaceted problems by handling complex geometries and boundaries. However, FEM is an approximate and basically, a numerical method. Nevertheless, it is sufficiently general and extremely effective in solving multi-physics problems. Although FEM originated as a break-through in solid mechanics, it has now infiltrated a wide range of disciplines including fluid mechanics, fatigue and fracture mechanics, thermal and electrical analysis, amongst many others. For structural or solid mechanics applications, there are many excellent books on FEM [1, 2]. The method has been successfully applied to beam and frame vibration problems [3, 4]. The basic building block in FEM is the so-called finite element, which for dynamic problems, possesses both the stiffness and mass properties of the element. The structure to be analysed is generally idealised as a collection of (finite) elements joined together, and FEM works on the premise that if the properties of all individual elements in the structure are known, the properties of the overall final structure can be worked out by assembling the individual properties of each element in the structure. The fundamental assumption based on which the FEM works is the assumption of shape functions which define the deformed shape of individual elements in the structure. Formulating a shape function requires some engineering judgement, which obviously, introduces approximation in the result. However, when the number of elements in FEM is increased, the result becomes more and more accurate and converges towards the exact result. Clearly, FEM is restricted to the total degrees of freedom in the structure. For instance, when carrying out a free vibration analysis using FEM, if the mass and stiffness matrices of a structure are each of the order of $n \times n$, there is no way, one can compute the $(n+1)$th natural frequency of the structure. Moreover, the higher order natural frequencies close to and including the nth natural frequency are expected to be inaccurate or even unreliable. This is a shortcoming of FEM, particularly when carrying out free vibration analysis in the high frequency range, which is required in the statistical energy analysis (SEA) method as the modal density in the high frequency range is generally very high [5, 6].

Against the above background, there is a powerful alternative to FEM in free vibration analysis which has superior modelling capability. The alternative is that of the dynamic stiffness method (DSM). It is based on the exact solution of the governing differential equation of the structural element when undergoing free natural vibration. There are of course, similarities between FEM and DSM. Both are based on the concept of shape functions and nodes of a structure. Notably, DSM uses the frequency-dependent exact shape functions obtained from the solution of the governing differential equation as opposed to the frequency-independent assumed shape functions used in FEM. The procedure to assemble properties of individual structural elements to form the overall matrix is essentially the same. However, there are some significant differences between FEM and DSM. For instance, when solving free vibration problems, the mass and stiffness matrices of individual elements are assembled separately in FEM to form the overall mass and stiffness matrices of the final structure. However, in DSM, there is only one frequency-dependent matrix called the dynamic stiffness matrix containing both the mass and stiffness properties of the element, which is assembled to form the overall dynamic stiffness matrix of the final structure. The other striking feature which distinguishes the two methods is the solution technique for the eigenvalue problem yielding the natural frequencies. FEM generally leads to a linear eigenvalue problem of the type $\mathbf{K}-\lambda\mathbf{M}=0$ where **K** and **M** are respectively the overall stiffness and mass matrices of the complete structure and $\lambda=\omega^2$ is the eigen parameter, ω being the circular or angular frequency. By contrast, DSM leads to a non-linear eigenvalue problem of the type $\mathbf{K_D}(\omega)=0$, where $\mathbf{K_D}(\omega)$ is the frequency dependent dynamic stiffness matrix of the complete structure. The best available solution technique to date, is the Wittrick-Williams algorithm [7] which is routinely used in DSM. As all the assumptions made in DSM are within the limits of the governing differential equations, the results from DSM are designated as exact and they are independent of the number of elements used in the analysis. Thus, unlike FEM, further discretization of a structure in DSM is not needed unless there is a change in the geometry or material properties. For instance, a single structural element can be used in DSM to compute any number of natural frequencies of a uniform beam to any desired accuracy, which of course, is impossible in FEM. Basically, DSM accounts for an infinite number of degrees of freedom of a freely vibrating structure whereas FEM being restricted to a selected number of degrees of freedom at the nodes, does not. For standard structures like beams and plates, DSM gives the same results as the classical theories based on governing differential equations. The purpose of this paper is to assess the accuracy and reliability of FEM in free vibration analysis of beams and frameworks, essentially by comparison with DSM.

2 Free vibration analysis of beams and frameworks using FEM

The basic building blocks in the free vibration analysis of beams and frameworks in FEM are the mass and stiffness matrices of individual elements which are derived using assumed shape functions as polynomials in terms of arbitrary constants. These matrices for beam elements are derived from separate consideration of axial and bending displacements. The procedure can be found in standard FEM texts [2, 8].

In a rectangular Cartesian coordinate system, Figure 1 shows a beam element of length L with its centroidal axis coinciding with the x-axis of the coordinate system. The origin is taken to be at the left-hand end at node 1 whereas the right-hand end at a distance L is at node 2 in the usual (customary) notation.

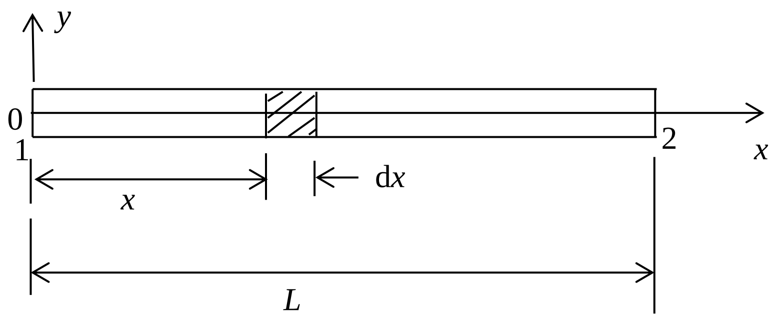

Figure 1. Coordinate system and notation for a beam element.

Now referring to Figure 2 which shows a beam element undergoing only axial deformations, with displacements δ_{x1} and δ_{x2} at nodes 1 and 2, respectively and the corresponding axial forces being f_{x1} and f_{x2}, respectively, the mass and stiffness matrices of the beam in axial motion, can be written as [2, 8]

$$\mathbf{m_A} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \qquad \mathbf{k_A} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1}$$

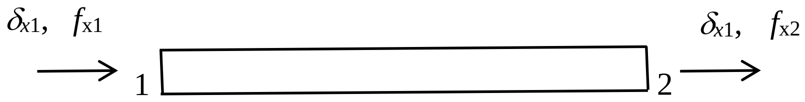

Figure 2. Nodal displacements and forces of a beam in axial deformation

where EA, ρA and L are the extensional (or axial) rigidity, mass per unit length and the length of the element, respectively.

For harmonic oscillation with angular or circular frequency ω rad/s, Equation (1) can be recast in the following form, relating the amplitudes of displacement vector $\boldsymbol{\delta}_\mathbf{A}$ and the force vector $\mathbf{f_A}$.

$$[\mathbf{k_A} - \omega^2 \mathbf{m_A}]\{\boldsymbol{\delta}_\mathbf{A}\} = \{\mathbf{f_A}\} \tag{2}$$

where $\boldsymbol{\delta}_\mathbf{A}$ is the displacement vector $\{\delta_{x1} \;\; \delta_{x2}\}^T$ and $\mathbf{f_A}$ is the force vector $\{f_{x1} \;\; f_{x2}\}^T$ of the beam element and the upper suffix T denoting a transpose.

For free vibration problems, the right-hand side of Equation (2), i.e. the force vector $\{\mathbf{f_A}\}$ will be zero.

Now referring to Figure 3 which shows the beam of Figure 1 undergoing flexural or bending displacement and bending rotation (δ_{y1}, θ_1) and (δ_{y2}, θ_2) at nodes 1 and 2, respectively with the corresponding forces and moments being (f_{y1}, m_1) and (f_{y2}, m_2), respectively, the mass and stiffness matrices of the beam in flexural or bending motion can be written as [2, 8]

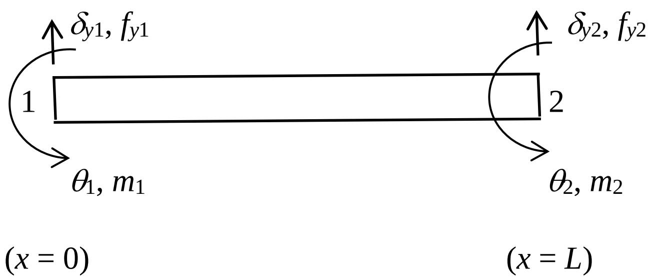

Figure 3. Nodal displacements and rotations of a beam in bending deformation.

$$\mathbf{m_B} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}; \quad \mathbf{k_B} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{8}$$

where ρA and EI are respectively the mass per unit length and bending rigidity, and L is the length of the beam.

For harmonic oscillation with angular or circular frequency ω rad/s, Equation (8) can be recast in the following form, relating the amplitudes of displacement vector $\boldsymbol{\delta}_\mathbf{B}$ and the force vector $\mathbf{f_B}$.

$$[\mathbf{k_B} - \omega^2 \mathbf{m_B}]\{\boldsymbol{\delta}_\mathbf{B}\} = \{\mathbf{f_B}\} \tag{9}$$

where $\boldsymbol{\delta}_\mathbf{B}$ is the displacement vector $\{\delta_{y1}\ \theta_1\ \delta_{y2}\ \theta_2\}^T$ and $\mathbf{f_B}$ is the force vector $\{f_{x1}\ m_1\ f_{x2}\ m_2\}^T$ of the beam element and T denotes a transpose.

For free vibration problems, the right-hand side of Equation (9), i.e. the force vector $\{\mathbf{f_B}\}$ will be zero.

The axial and bending mass and stiffness matrices of the beam given by Equations. (1) and (8) can be combined with the help of Figure 4 to give the mass and stiffness matrices of the beam undergoing combined axial and bending deformations, i.e., the beam has three degrees of freedom (δ_x, δ_y and θ) at each node, as follows [2, 8]

$$\mathbf{m} = \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ 0 & b_1 & b_3 & 0 & b_4 & -b_5 \\ 0 & b_3 & b_2 & 0 & b_5 & b_6 \\ a_2 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -b_4 & b_5 & 0 & b_1 & -b_3 \\ 0 & -b_5 & b_6 & 0 & -b_3 & b_2 \end{bmatrix} \tag{10}$$

where

$a_1 = \rho AL/3$; $a_2 = \rho AL/6$; $b_1 = 13\rho AL/35$; $b_2 = \rho AL^3/105$;
$b_3 = 11\rho AL^2/210$; $b_4 = 9\rho AL/70$; $b_5 = 13\rho AL^2/420$; $b_6 = \rho AL^3/140$ (11)

and

$$\mathbf{k} = \begin{bmatrix} c_1 & 0 & 0 & -c_1 & 0 & 0 \\ 0 & d_1 & d_3 & 0 & -d_1 & d_3 \\ 0 & d_3 & d_2 & 0 & -d_3 & d_4 \\ -c_1 & 0 & 0 & c_1 & 0 & 0 \\ 0 & -d_1 & -d_3 & 0 & d_1 & -d_3 \\ 0 & d_3 & d_4 & 0 & -d_3 & d_2 \end{bmatrix} \tag{12}$$

where

$$c_1 = EA/L; \quad d_1 = 12EI/L^3; \quad d_2 = 4\mathrm{EI/L}; \quad d_3 = 6\mathrm{EI/L}^2; \quad d_4 = 2EI/L \tag{13}$$

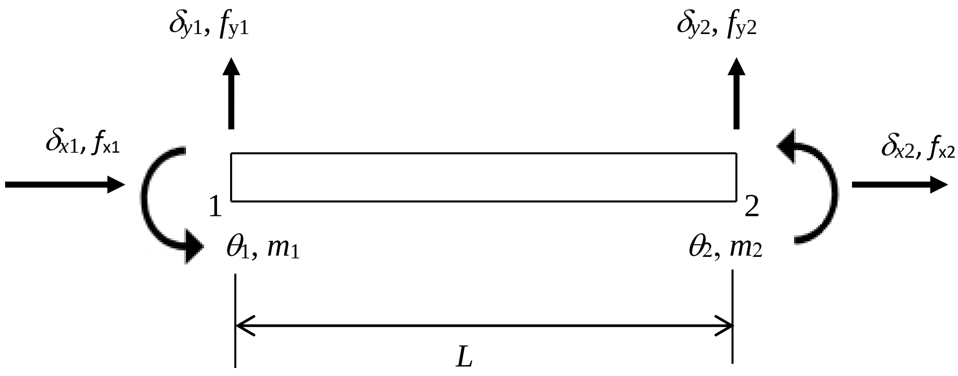

Figure 4. Nodal displacements and forces of a beam with axial and bending deformations.

The free vibration problem for a beam with three degrees of freedom (δ_x, δ_y and θ) at each node can be formulated with the help of Equations (10) and (12), as

$$[\mathbf{k} - \omega^2\mathbf{m}]\{\boldsymbol{\delta}\} = 0 \tag{14}$$

where $\boldsymbol{\delta}$ is the displacement vector $\{\delta_{x1}\ \delta_{y1}\ \theta_1\ \delta_{x2},\ \delta_{y2}\ \theta_2\}^T$ for a free-free beam.

For free vibration analysis of plane frames, the mass and stiffness matrices of each beam member in the frame given by Equations. (10) and (12) must be assembled with transformation from local to global co-ordinates so forming the overall mass (**M**) and stiffness (**K**) matrices of the final structure to formulate the linear eigenvalue problem

$$[\mathbf{K} - \lambda\mathbf{M}]\{\boldsymbol{\Delta}\} = 0 \tag{15}$$

where $\lambda = \omega^2$ is the eigen-parameter to be computed for non-trivial solution of $\boldsymbol{\Delta}$, yielding the natural frequencies of the frame.

Thus, to arrive at the eigenvalue problem of Eq. (15), the element mass and stiffness matrices of all beam elements in a frame must be transformed from their local coordinates to the global (or datum) coordinate system and then assembled [2, 8] to form the overall mass (**M**) and stiffness (**K**) matrix of the complete structure.

Figure 5 shows that the local (*xy*) and global (XY) coordinate systems of a beam, with the local *x*-axis, making an angle β relative to the global (or datum) X-axis in the anti-clockwise direction.

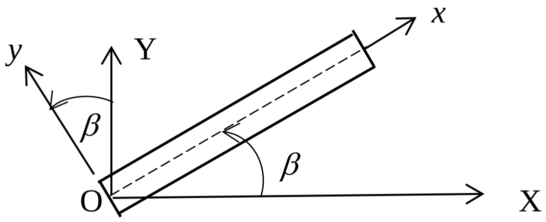

Figure 5. Local and global coordinate systems of a beam element (*xy*: Local coordinates; XY: Global coordinates)

For the local and global coordinate systems shown in Figure 5, the transformation matrix $\mathbf{T}$ required to transform the mass and stiffness matrices of Equations. (10) and (12) is given by [2, 8]

$$\mathbf{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

where

$$C = \cos\beta; \quad S = \sin\beta \tag{17}$$

The transformed mass and stiffness matrices from local to global coordinates are given by [2, 8]

$$\bar{\mathbf{m}} = \mathbf{T}^T\mathbf{m}\mathbf{T}; \qquad \bar{\mathbf{k}} = \mathbf{T}^T\mathbf{k}\mathbf{T} \tag{18}$$

3 Free vibration analysis of beams and frameworks using DSM

Unlike the finite element method (FEM), the dynamic stiffness method (DSM) is essentially an exact method which gives the same result as the classical approach of solution of the governing differential equation exactly. For a given structure, the conditions of equilibrium usually formulated for nodes are in fact conditions of a unique exact solution when the DSM is applied. This is in sharp contrast to FEM. In some other ways, such as the element assembly process, DSM is analogous to FEM. Nevertheless, there are significant differences between the two methods. For instance, as indicated earlier, when carrying out the free vibration analysis of a frame, FEM uses separate mass and stiffness matrices of individual elements and assembles them to form the overall mass and stiffness matrices of the frame. By contrast, the DSM uses only one frequency-dependent matrix of each individual element, which contains both the mass and stiffness properties of the element, and then assembles the dynamic stiffness matrices of all elements to form the overall dynamic stiffness matrix of the frame in the same way as FEM. The transformation matrix of Equation (16) can be used to transform the dynamic stiffness matrix of individual beam elements from local to global coordinates during the assembly process. The other main difference between FEM and DSM is, of course, in the solution technique for which the FEM generally leads to a linear eigenvalue problem whereas the DSM formulation generates a nonlinear eigenvalue problem requiring the application of the Wittrick-Williams algorithm [7]. The element dynamic stiffness matrices of a beam element in axial and flexural motions are available in the literature [9, 10], and they are summarised below.

Referring to Figure 2, the dynamic stiffness matrix of a beam element in axial motion which relates the amplitudes of axial forces to those of the displacements at nodes can be written as [9, 10]

$$\mathbf{k}_{\mathbf{D}}^{\mathbf{A}}(\omega) = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix} \tag{19}$$

where

$$a_1 = \frac{EA}{L}\mu\cot\mu\,; \quad a_2 = -\frac{EA}{L}\mu\mathrm{cosec}\mu \tag{20}$$

with

$$\mu = \sqrt{\frac{\rho A\omega^2 L^2}{EA}} \tag{21}$$

and EA and L which are axial or extensional rigidity and length of the beam as defined previously.

Similarly, referring to Figure 3, the frequency-dependent dynamic stiffness matrix of the beam in Figure 1 in bending motion can be written as [9, 10]

$$\mathbf{k_D^B}(\omega) = \begin{bmatrix} d_1 & d_2 & d_4 & d_5 \\ d_2 & d_3 & -d_5 & d_6 \\ d_4 & -d_5 & d_1 & -d_2 \\ d_5 & d_6 & -d_2 & d_3 \end{bmatrix} \tag{22}$$

where

$$d_1 = W_3\lambda^3(S_\lambda C_{h\lambda} + C_\lambda S_{h\lambda})/\Delta,\;\; d_2 = W_2\lambda^2 S_\lambda S_{h\lambda}/\Delta,\;\; d_3 = W_1\lambda(S_\lambda C_{h\lambda} - C_\lambda S_{h\lambda})/\Delta \tag{23}$$

$$d_4 = -W_3\lambda^3(S_\lambda + S_{h\lambda})/\Delta,\;\; d_5 = W_2\lambda^2(C_{h\lambda} - C_\lambda)/\Delta,\;\; d_6 = W_1\lambda(S_{h\lambda} - S_\lambda)/\Delta \tag{24}$$

with

$$W_1 = \frac{EI}{L};\; W_2 = \frac{EI}{L^2};\; W_3 = \frac{EI}{L^3};\lambda = \sqrt[4]{\frac{\rho A\omega^2 L^4}{EI}} \tag{25}$$

$$C_\lambda = \cos\lambda\,;\; S_\lambda = \sin\lambda\,;\; C_{h\lambda} = \cosh\lambda\,;\; S_{h\lambda} = \sinh\lambda \tag{26}$$

and

$$\Delta = 1 - C_\lambda C_{h\lambda} \tag{24}$$

Now the axial and bending dynamic stiffness matrices of Equations (19) and (22) can be combined using the notation of Figure 4 to give the 6×6 dynamic stiffness $\mathbf{k_D}(\omega)$ of a beam which contains both the axial and bending components of the amplitudes of forces and displacements at each node so that it can be implemented to solve the free vibration problem of a plane frame. Thus, $\mathbf{k_D}(\omega)$ is given by [10]

$$\mathbf{k_D}(\omega) = \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ 0 & d_1 & d_2 & 0 & d_4 & d_5 \\ 0 & d_2 & d_3 & 0 & -d_5 & d_6 \\ a_2 & 0 & 0 & a_1 & 0 & 0 \\ 0 & d_4 & -d_5 & 0 & d_1 & -d_2 \\ 0 & d_5 & d_6 & 0 & -d_2 & d_3 \end{bmatrix} \tag{25}$$

When analysing a framework for its free vibration characteristics, the above dynamic stiffness matrix of all individual beam elements in the frame must be assembled using the same procedure as FEM to form the overall dynamic stiffness matrix $\mathbf{K_D}(\omega)$ of the complete frame. The transformation matrix, given by Equation (16) can be used to assemble the element dynamic stiffness matrices, noting that there is only one matrix for each individual element to transform from local to global coordinates rather than separate mass and stiffness matrices, as required in FEM. Once the overall dynamic stiffness matrix $\mathbf{K_D}(\omega)$ is formed, the following eigenvalue problem is solved by using the Wittrick-Williams algorithm [7].

$$\mathbf{K_D}(\omega)\mathbf{\Delta} = 0 \tag{26}$$

where $\mathbf{\Delta}$ represents the amplitudes of the displacement vector of the frame.

The Wittrick-Williams algorithm [7], which has featured in literally hundreds of papers, is applied to Equation (26) to extract the natural frequencies of the frame to any desired accuracy. The computer implementation of the algorithm is simple and straightforward. Basically, the algorithm monitors the Sturm sequency property of the dynamic stiffness matrix and ascertains the natural frequencies of a structure with certainty, making sure that none of its natural frequencies is missed. The details of the Wittrick-Williams algorithm are not elaborated here because of its extensive coverage in the literature [11, 12].

4 Results and discussion

4.1. Free axial or longitudinal vibration of a cantilever beam

The first illustrative example is a uniform cantilever beam undergoing free axial or longitudinal vibration, by using a 2-element idealisation for FEM application, as shown in Figure 6. The axial rigidity of the beam in the usual notation is EA and its mass per unit length is ρA. Each of the two elements has a length L_1 so that the total length L of the beam is $2L_1$. The nodes 1, 2 and 3 are numbered from the built-in end to the free end as shown.

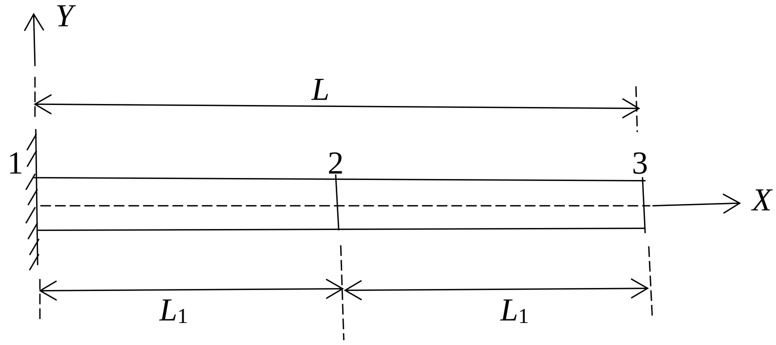

Figure 6. Two-element idealisation of a beam element in axial vibration

If only axial deformation is considered, i.e., by ignoring the bending displacement and bending rotation, the element mass and stiffness matrices of elements connecting nodes 1-2 and 2-3 can be assembled with the help of Equation (1) and then removing the row and column of the overall mass and stiffness matrices corresponding to the zero displacement at the built-in end (i.e., at node 1) to give:

$$\mathbf{M} = \frac{\rho A L_1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A L}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}; \mathbf{K} = \frac{EA}{L_1}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \frac{2EA}{L}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \tag{27}$$

Now, formulating the eigenvalue problem yields the following determinantal (frequency) equation

$$|\mathbf{K} - \omega^2\mathbf{M}|=0 \tag{28}$$

Substituting **K** and **M** from Equation (27) into Equation (28) and expanding the determinant yield the following frequency equation:

$$7\lambda^2 - 10\lambda + 1 = 0 \tag{29}$$

where

$$\lambda = \frac{\rho A L^2 \omega^2}{24EA} \tag{30}$$

As the mass and stiffness matrices each have order 2×2, only two natural frequencies can be extracted from the 2-element FEM idealisation of the beam shown in Figure 6. As expected, the resulting frequency equation is quadratic, see Equation (29).

The two roots of λ from Equation (29) with the help of Equation (30) yield the following two natural frequencies of the beam in axial or longitudinal vibration when using the 2-element idealisation of Figure 6.

$$\omega_1 = 1.6114\sqrt{\frac{EA}{\rho A L^2}}; \qquad \omega_2 = 5.6293\sqrt{\frac{EA}{\rho A L^2}} \tag{31}$$

Using DSM, which gives the same results as the classical approach of solving the governing differential equations, the above two frequencies are shown below:

$$\omega_1 = \frac{\pi}{2}\sqrt{\frac{EA}{\rho A L^2}} \simeq 1.5708\sqrt{\frac{EA}{\rho A L^2}}; \qquad \omega_2 = \frac{3\pi}{2}\sqrt{\frac{EA}{\rho A L^2}} \simeq 4.7124\sqrt{\frac{EA}{\rho A L^2}} \tag{32}$$

The errors in the FEM results for the above two frequencies are 2.58% and 19.46%, respectively. The error in the second natural frequency is much larger, as expected.

Clearly, no more than two natural frequencies can be extracted from the 2-element idealisation using FEM whereas the DSM does not need any further discretisation to improve the results because a single element in DSM is sufficient to determine any number of the natural frequencies of the beam to any desired accuracy.

Table 1 shows the first 10 natural frequencies of the beam in free axial or longitudinal vibration in non-dimensional form, using the number of elements N as 2, 4, 8 and 10 when using FEM together with the exact results obtained from DSM by using only one element (further verified by the classical theory of differential equation). As can be seen in Table 1, the first two natural frequencies of the beam can be predicted reasonably accurately using 4 or more elements in FEM, but the inaccuracy rapidly builds up when predicting the higher order natural frequencies. With 10 elements in FEM, the first five natural frequencies of the beam are within 10% of the DSM results, but beyond the 5th natural frequency, the errors are above 10%, e.g., the error in the 10th natural frequency is around 15%. The errors are expected to be still higher beyond the 10th natural frequency. When using the statistical energy analysis method [5, 6] which requires modal densities in the high frequency range, such errors may not be acceptable.

Frequency Number (i)	Natural frequency $\omega_i\sqrt{\rho A L^2/EA}$				
	FEM results using N elements				DSM results
	N=2	N=4	N=8	N=10	
1	1.6114	1.5809	1.5733	1.5724	1.5708
2	5.6293	4.9872	4.7808	4.7561	4.7124
3		9.0594	8.1719	8.0571	7.8540
4		13.101	11.866	11.554	10.996
5			15.946	15.320	14.137
6			20.336	19.400	17.279
7			24.533	23.754	20.420
8			27.318	28.145	23.562
9				31.986	26.704
10				34.324	29.845

Table 1. Natural frequencies of a beam in axial vibration using FEM and DSM

4.2. Free flexural or bending vibration of a simply-supported beam

The next example is the simply supported beam shown in Figure 7 undergoing free bending or flexural vibration. The beam is idealised in FEM using only one-element connecting nodes 1 and 2.

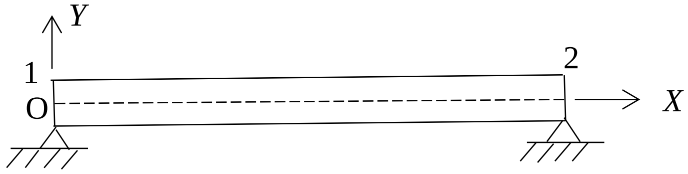

Figure 7. One-element idealisation of a simply supported beam using FEM.

Using the mass and stiffness matrices of a beam element given by Equation (8) and applying the boundary conditions of Figure 7, i.e. by deleting the rows and columns of the mass and stiffness matrices corresponding to zero displacements (Δ_X and Δ_Y) at each node, the following determinant of the frequency equation is obtained.

$$|\mathbf{K}-\omega^2\mathbf{M}| = \left|\frac{EI}{L^3}\begin{bmatrix}4L^2 & 2L^2\\ 2L^2 & 4L^2\end{bmatrix} - \frac{\omega^2\rho AL}{420}\begin{bmatrix}4L^2 & -3L^2\\ -3L^2 & 4L^2\end{bmatrix}\right| = 0 \tag{33}$$

or

$$7\lambda^2 - 44\lambda + 12 = 0 \tag{34}$$

where

$$\lambda = \frac{\rho AL^4\omega^2}{420EI} \tag{35}$$

Equation (34) gives

$$\lambda_1 = \frac{2}{7}; \qquad \lambda_2 = 6 \tag{36}$$

Substituting λ_1 and λ_2 from Equation (36) into Equation (35) yields

$$\omega_1 = 10.954\sqrt{\frac{EI}{\rho AL^4}}; \quad \omega_2 = 50.200\sqrt{\frac{EI}{\rho AL^4}} \tag{37}$$

The exact expression for the nth natural frequency ω_n of a simply-supported beam is available in the following closed analytical form [9].

$$\omega_n = (n\pi)^2\sqrt{\frac{EI}{\rho AL^4}} \tag{38}$$

The errors in the two natural frequencies using FEM based on one-element idealisation are around 11% and 27%, respectively. Of course, the errors will diminish with increasing number of elements (N) used in FEM. Table 2 shows the first ten natural frequencies of a simply-supported beam using 2, 4, 8 and 10 elements alongside the exact results obtained by DSM with one element. Large errors in FEM analysis when computing higher natural frequencies with fewer elements are evident.

Frequency Number (i)	Natural frequency $\omega_i\sqrt{\rho AL^4/EI}$				
	FEM results using N elements				DSM results
	N=2	N=4	N=8	N=10	
1	9.9086	9.8722	9.8698	9.8697	9.8696
2	43.818	39.634	39.489	39.483	39.478
3	110.14	90.450	88.941	88.874	88.826
4	200.80	175.27	158.54	158.18	157.91
5		278.59	249.03	247.71	246.74
6		440.56	361.80	358.13	355.31
7		660.02	498.67	490.47	483.61
8		803.19	701.08	646.21	631.65
9			872.87.	826.48	799.44
10			1114.4	1095.4	986.96

Table 2. Natural frequencies of a simply supported beam in flexural vibration using FEM and DSM

4.3. Free vibration of a plane frame

The final set of results was obtained for a plane frame with its geometrical dimensions shown in Figure 8, by using both FEM and DSM. In the DSM analysis, each member of the frame was a single beam element so that altogether 13 elements are used whereas in the FEM analysis, two models, namely Model A and Model B were used. Model A is treated in the same way as DSM, i.e. each member of the frame is a single beam element, but in Model B, each member of the frame was split into two beam elements of equal length so that altogether 26 elements are used. The properties of each beam element in the frame are EA= 8×10^8 N, EI= 4×10^6 Nm^2, ρA= 30 kg/m.

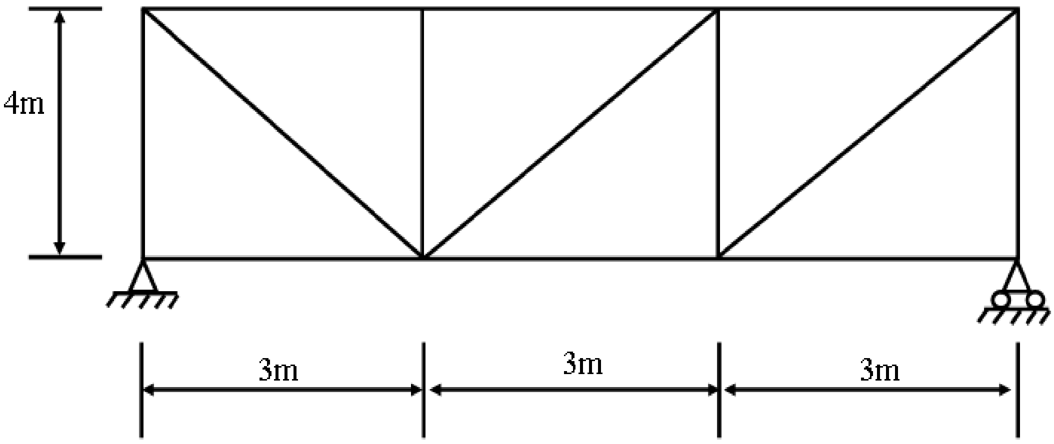

Figure 8. A plane frame for free vibration analysis using FEM and DSM.

The first five natural frequencies of the frame shown in Figure 8 were computed using DSM, and using Model-A and Model-B of FEM and the results are shown in Table 3 with the percentage errors shown in parenthesis.

Natural frequency ω_i (rad/s)	FEM Results (rad/s)		DSM Results (rad/s)
	Model-A result (%error)	Model-B result (%error)	
ω_1	267.39 (18.97%)	226.47 (0.76%)	224.76
ω_2	330.70 (34.60%)	248.07 (0.97%)	245.69
ω_3	336.06 (25.69%)	270.11 (1.02%)	267.38
ω_4	389.32 (20.57%)	325.34 (0.75%)	322.92
ω_5	413.24 (21.94%)	341.71 (0.84%)	338.88

Table 3. Natural frequencies of a frame using FEM and DSM

Clearly, Model-A, in which each member of the frame was modelled as a single beam in the FEM analysis, gives large errors in all the natural frequencies when compared to DSM results, but by doubling the number of elements in the FEM analysis, the errors can be reduced to around 1%.

4 Conclusions

The accuracy and reliability of the finite element method (FEM) in free vibration analysis of beams and frames are assessed by using the dynamic stiffness method. Representative results are given for individual beams and a framework. Errors in FEM analysis, particularly in the high frequency range, are evident. This can be a disadvantage when applying FEM to the statistical energy analysis method which requires accurate and reliable modal analysis in the high frequency range. Similar analyses can be carried out on complex structures comprising beams, plates, and shells.

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References

[1] O.C. Zienkiewicz, R.L. Taylor, J.Z. Zhu, "The Finite Element Method: Its Basis and Fundamentals", 7th edition, Elsevier, 2013.

[2] K.J. Bathe, "Finite Element Procedures in Engineering Analysis", Prentice-Hall, 1982.

[3] S.W. Lee, Y.H. Kim, "A New Approach to the Finite Element Modelling of Beams with Warping Effect", International Journal for Numerical Methods in Engineering, 24(12), 2327-2341, 1987.

[4] G. Wei, P. Lardeur, F. Druesne, "A New Solid-Beam Approach Based on First or Higher-Order Beam Theories for Finite Element Analysis of Thin to Thick Structures". Finite Elements in Analysis and Design, 200, 2022, Paper 103655.

[5] R.H. Lyon, R.G. DeJong, "Theory and Application of Statistical Energy Analysis", second edition. London, UK: Butterworth-Heinermann; 1995.

[6] A.J. Keane, W.G. Price, "Statistical Energy Analysis: An Overview, with Applications in Structural Dynamics", Cambridge University Press, UK, 2008.

[7] W.H. Wittrick, F.W. Williams, "A General Algorithm for Computing Natural Frequencies of Elastic Structures", Quarterly Journal of Mechanics and Applied Mathematics, 24(3), 263-284, 1971.

[8] J.S. Przemieniecki, "Theory of Matrix Structural Analysis", Dover Publications Inc; 1985.

[9] V. Kolousek, "Dynamics in Engineering Structures", Butterworth, London, 1973.

[10] F.W. Williams, W.P. Howson "Compact Computation of Natural Frequencies and Buckling Loads for Plane Frames", International Journal for Numerical Methods in Engineering, 11, 1067-1081, 1977.

[11] F.W. Williams, W.H. Wittrick, "Exact Buckling and Frequency Calculations Surveyed", Journal of Structural Engineering, 109, 109-187, 1983.

[12] J.R. Banerjee, "Review of the Dynamic Stiffness Method for Free Vibration Analysis of Beams", Transport Safety and Environment, 1(2), 106-116, 2019.