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Combined Random and Deterministic Effects in a Simple Aeroelastic Model

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Abstract

The response of slender engineered structures in close proximity to the lock-in frequency region exhibits multiple dominant frequencies that contribute to the quasi-periodic nature of the response. The difference in individual dominant frequencies increases significantly with increasing distance from the lock-in region. This effect alters the character of the response from apparently non-stationary to quasi-periodic, with the frequency of beating varying as the distance from the locking interval changes. In the presence of combined random and harmonic excitation, the response character varies between stationary, cyclo-stationary, and non-stationary, depending on the intensity of the stochastic component.

While the probabilistic characteristics of a non-linear Single Degree of Freedom oscillator of the van der Pol type system on a slow time scale can be described using partial amplitudes of the response, this paper specifically focuses on the non-stationary case. The solution to the Fokker-Planck equation for the cross-Probability Density Function of the partial amplitudes is determined using the Galerkin approximation. For this purpose, orthogonal polynomial basis functions are utilized and assessed.

Keywords: Fokker-Planck equation, stochastic averaging, numerical solution, Galerkin approximation, van der Pol-type oscillator, partial amplitudes

1 Introduction

Large and slender engineering structures, including bridge decks, masts, high-rise buildings, or power lines and catenary systems, are prone to diverse vibration effects due to their inherent softness. Considering the underlying physics, these structures exhibit various non-linear modes influenced by the ratio or difference between natural and excitation frequencies, even in simple cases. In the vicinity of the lock-in region, the non-stationary response takes on a quasi-periodic nature, with the beating frequency significantly varying with distance from the lock-in frequency interval. This distinctive behaviour is notably influenced by the presence of combined random and harmonic excitation.

Analogous phenomena are observed in many non-linear physical problems, where synchronization during resonance is simultaneously counteracted by random disturbances. For instance, wind-induced vibration caused by vortex shedding interacts with transport-induced excitation, [1]. Similarly, situations arise where multiple wind-induced excitation modes coincide due to changes in the structure's geometry, such as disturbances caused by icing, [2].

The stochastic characterization of the response is determined by the solution of the Fokker-Planck equation (FPE), which, in such a general case, is time-dependent. The previously employed approach, based on the formulation of partial response amplitudes, only permits the solution of a simplified problem in which the high-frequency components are averaged. The probability density function (PDF) obtained in this manner describes the averaged stationary response characteristics of the system. Whether such a solution is sufficient depends on the particular problem.

Various procedures exist for solving the Fokker-Planck equation (FPE). For numerical solutions based on finite differences or finite elements, refer to works such as [3] or the comprehensive review in [4]. The sophisticated approximate methods with specific conditions for their application are subject of research for a long time, see, e.g., review paper [5]. In particular, the usually employed methods comprise the Perturbation Method [6], which is effective for systems with weak nonlinearity or the Stochastic Averaging Method [7, 8] or Gaussian or non-Gassian Closure Methods [9, 10]. Negative values in PDF tail regions may occur with the Gram-Charlier series method [11]. The widely used Equivalent Linearization (EQL) Method [12] is effective only for weakly nonlinear systems, assuming a Gaussian distribution for the PDF solution. Very promissive is the Exponential Polynomial Closure (EPC) Method [13–15] has proven effective for studying stationary PDFs, even with strong nonlinearities. The recent references offer modifications for analyzing non-stationary probabilistic solutions of nonlinear oscillators.

The present work by the authors builds on results derived previously for the resonant stationary case, [2], specifically focusing on the response of the van der Pol-type single-degree-of-freedom (SDOF) oscillator with additive excitation that combines deterministic and random components in the lock-in region, i.e., synchronized eigen- and vortex-shedding frequencies. In this paper, the complete evolutionary Fokker-

Planck equation (FPE) is utilized, ruling out the possibility of finding an exact solution. Knowledge of the exact solution of the corresponding FPE for the case of zero detuning $\Delta = 0$ is exploited, which is then refined for $\Delta \neq 0$ in the form of a Galerkin approximation. The coefficients of this approximation are determined using the Galerkin-Petrov orthogonalization procedure. The refinement is based on polynomial basis functions.

This conference contribution presents intermediate findings from an ongoing research project aimed at determining the time-dependent PDF of a non-stationary response. Emphasis is placed on understanding the complex dynamics exhibited by a nonlinear SDOF oscillator under combined harmonic and random excitations. These results are offered as a reflection of the current progress, representing a working paper that is part of the broader efforts to achieve a more precise time estimation in subsequent phases of this research.

2 Mathematical model

The single-degree-of-freedom (SDOF) oscillator of van der Pol type, frequently employed to describe transverse wind-generated vibrations under additive excitation that merges deterministic and random components, is represented by Eq. (1a). In in such a configuration, the trivial solution becomes unstable, while the limit cycle attains stability. Consequently, this model can capture beating effects, particularly for large detuning of the linear natural frequency and the harmonic excitation component. Additionally, it reflects the stabilization of the lock-in response due to the presence of a stable limit cycle. The governing differential equation and its normal form are given as

$$\ddot{u} - (\eta - \nu u^2)\dot{u} + \omega_0^2 u = f(t)\,, \tag{1a}$$

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= (\eta - \nu u^2)v - \omega_0^2 u + P\omega^2 \cos \omega t + h\xi(t). \end{aligned} \tag{1b}$$

where:

u, v is the displacement $[m]$ and velocity $[ms^{-1}]$;

η, ν are the parameters of the linear and quadratic damping, respectively $[s^{-1}, s^{-1}m^{-2}]$;

ω_0, ω are the eigen-frequency of the linear SDOF system and frequency of the vortex shedding $[s^{-1}]$;

$f(t)$ represents external excitation: $f(t) = P\omega^2 \cos \omega t + h\xi(t)$;

$P\omega^2$ and $\xi(t)$ are the amplitude of the harmonic excitation force $[ms^{-2}]$ and the broad-band Gaussian random process $[1]$; and

h is the multiplicative constant $[ms^{-2}]$.

In the following, it will be supposed that both input and output processes are Markovian. The general problem can be reformulated as a system of n first-order Itô stochastic differential equations, with N random noise terms in a normal form, expressed using Einstein notation as follows:

$$\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} = f_j(\mathbf{x},t) + g_{jr}(\mathbf{x},t)w_r(t)\ , \quad j = 1,\ldots,n; \;\; r = 1,\ldots,N \tag{2}$$

for $n = 2, N = 1$, and

$\mathbf{x} = (x_1, x_2)^T = (u, \dot{u})^T$ denote the response partial amplitudes;

$f_j(\mathbf{x},t), g_{jr}(\mathbf{x},t)$ are continuous deterministic functions of the state variables $\mathbf{x}$ and time t;

$w_r(t) = w_1(t) = \xi(t)$ is the covariance-stationary random noise with zero, possessing the characteristics of a Markov process; in general, is this process not delta-correlated.

The response PDF is then governed by the FPE:

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial x_j}\left(\kappa_j(\mathbf{x},t)\cdot p(\mathbf{x},t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_k}(\kappa_{jk}(\mathbf{x},t)\cdot p(\mathbf{x},t)) \tag{3}$$

Parameters $\kappa_j(\mathbf{x},t)$ and $\kappa_{jk}(\mathbf{x},t)$ represent the first and second derivative moments, generally referred to as drift and diffusion coefficients, respectively. If the input random processes are covariance stationary and ergodic, then their correlation functions depend only on the difference $\tau = t_1 - t_2$, and parameters $\kappa_j(\mathbf{x},t)$ and $\kappa_{jk}(\mathbf{x},t)$ can be expressed as:

$$\kappa_j(\mathbf{x}_t,t) = f_j(\mathbf{x}_t,t) + \int_{-\infty}^{0} g_{ls}(\mathbf{x}_{t+\tau}, t+\tau)\frac{\partial}{\partial x_l} g_{j,r}(\mathbf{x}_t,t) R_{rs}\mathrm{d}\tau\ , \tag{4a}$$

$$\kappa_{jk}(\mathbf{x}_t,t) = g_{jr}(\mathbf{x}_t,t)\int_{-\infty}^{\infty} g_{kr}(\mathbf{x}_{t+\tau}, t+\tau) R_{rr}(\tau)\mathrm{d}\tau \tag{4b}$$

$$j,k,l = 1,\ldots,n, \quad r,s = 1,\ldots,N$$

where $R_{rr}(\tau)$ is the auto-correlation function of the input process $w_r(t)$. In the present case, only one process acts, so that $w_r(t) = \xi(t)$, and, consequently, $R_{rr}(\tau) = R(\tau)$.

The determination of the response PDF is addressed in two scenarios: (i) when the eigen-frequency ω_0 nearly matches the vortex-shedding frequency ω_s with only a small detuning $\Delta = |\omega_0 - \omega_s| < \Delta_l$ for some limit value Δ_l, leading to a stationary response and a time-independent PDF; (ii) when both frequencies significantly differ, $\Delta > \Delta_l$.

In the first case, the is treated as homogeneous, resulting in a reduced FPE. Stochastic averaging can be applied due to small-order terms, facilitating a solution in phase space coordinates without expecting beating effects.

In contrast, if the detuning $\Delta > \Delta_l$ is substantial (case ii), the solution to the FPE is non-stationary. Retaining the left-hand side of the FPE is necessary, and stochastic averaging is not applicable as it would eliminate the time-dependent processes characterizing the non-stationary response.

2.1 Stationary case

In the stationary case, the response is characterized by a limit cycle, and the stochastic averaging method, [7, 16], may be applied to obtain the averaged Itô system. Subsequently, the explicit solution to the reduced FPE can be expressed using a stationary potential; refer to [17] for details. To achieve this, the displacement and velocity variables may be expressed in trigonometric form as follows:

$$\begin{aligned} u(t) &= a_c \cos\omega t + a_s \sin\omega t\,, \\ v(t) &= -a_c\omega \sin\omega t + a_s\omega \cos\omega t\,, \end{aligned} \qquad \dot{a}_c \cos\omega t + \dot{a}_s \sin\omega t = 0\,.$$

Substitution into the original van der Pol equation gives:

$$\begin{aligned} \dot{a}_c =& 2\Delta \sin\omega t(a_c \cos\omega t + a_s \sin\omega t) - P\omega \sin\omega t \cos\omega t - \frac{h}{\omega}\sin\omega t \cdot \xi(t) \\ & - \sin\omega t[\eta - \nu(a_c \cos\omega t + a_s \sin\omega t)^2](-a_c \sin\omega t + a_s \cos\omega t), \end{aligned} \tag{5a}$$

$$\begin{aligned} \dot{a}_s =& -2\Delta\ \cos\omega t(a_c \cos\omega t + a_s \sin\omega t) + P\omega \cos^2\omega t + \frac{h}{\omega}\cos\omega t \cdot \xi(t) \\ & + \cos\omega t[\eta - \nu(a_c \cos\omega t + a_s \sin\omega t)^2](-a_c \sin\omega t + a_s \cos\omega t)\,. \end{aligned} \tag{5b}$$

where $\Delta = \dfrac{\omega_0^2 - \omega^2}{2\omega}$ is the frequency detuning.

The result of the time averaging over a period 2π (see [18–20] for mathematical details, and [16] for the engineering approach) can be symbolically written as

$$\mathrm{d}a_c = D_c \mathrm{d}t + \sigma_{cc}\mathrm{d}B_c\,, \qquad \mathrm{d}a_s = D_s \mathrm{d}t + \sigma_{ss}\mathrm{d}B_s \tag{6}$$

where D_c, D_c are the averaged deterministic parts, σ_{cc}, σ_{ss} denote the parameters of the spectral density $\Phi_{\xi\xi}(\omega)$, and $B_{c,s}(t)$ stands for the Wiener process corresponding to input excitation $\xi(t)$. In the further text, the averaged variables will be denoted using the same symbols because no confusion can occur, i.e., $a_s = \langle a_s \rangle$, where $\langle \bullet \rangle$ represents the averaging operator.

The spectral density of the Gaussian random noise in Eq. (1b) is generally concentrated around a single dominant frequency ω, the process is not white. Then the parameters σ_{cc}, σ_{ss} in Eq. (6) are given as

$$\sigma_{cc}^2 = \sigma_{ss}^2 = 2\pi\Phi_{\xi\xi}(\omega) \tag{7}$$

where $\Phi_{\xi\xi}(\omega)$ is the spectral density of the process $\xi(t)$ at frequency ω.

Finally, the the averaged Itô system reads:

$$\begin{aligned} \mathrm{d}a_c =& \frac{\pi}{\omega}\left[\ \eta a_c + 2\Delta a_s - \frac{1}{4}\nu \cdot a_c(a_c^2 + a_s^2)\right]\mathrm{d}t + \left(\frac{\pi}{\omega}\Phi_{\xi\xi}\right)^{\frac{1}{2}}\mathrm{d}B_c\,, \\ \mathrm{d}a_s =& \frac{\pi}{\omega}\left[-2\Delta a_c + \eta a_s - \frac{1}{4}\nu \cdot a_s(a_c^2 + a_s^2)\right]\mathrm{d}t + \frac{\pi}{\omega}P\omega\ \mathrm{d}t + \left(\frac{\pi}{\omega}\Phi_{\xi\xi}\right)^{\frac{1}{2}}\mathrm{d}B_s\,. \end{aligned} \tag{8}$$

2.1.1 No detuning considered, $\Delta = 0$

If the stationary solution exists, the partial amplitudes a_c and a_s represent stationary values and the time derivative in the FPE vanishes. As a result, only reduced FPE is to be solved:

$$\begin{aligned}
&\frac{\partial}{\partial a_c}\left(\left[\eta a_c + 2\Delta a_s - \frac{1}{4}\nu \cdot a_c(a_c^2 + a_s^2)\right] p\right) - \frac{1}{2\omega}\Phi_{\xi\xi}(\omega)\frac{\partial^2 p}{\partial a_c^2} \\
&+\frac{\partial}{\partial a_s}\left(\left[\eta a_s - 2\Delta a_c - \frac{1}{4}\nu \cdot a_s(a_c^2 + a_s^2) + P\omega\right] p\right) - \frac{1}{2\omega}\Phi_{\xi\xi}(\omega)\frac{\partial^2 p}{\partial a_s^2} = 0
\end{aligned} \tag{9}$$

together with the boundary conditions:

$$\lim_{|a_c|+|a_s|\to\infty}\left[\eta a_c + 2\Delta a_s - \frac{1}{4}\nu \cdot a_c(a_c^2 + a_s^2)\right] p - \frac{1}{2\omega}\Phi_{\xi\xi}\frac{\partial p}{\partial a_c} = 0, \tag{10a}$$

$$\lim_{|a_c|+|a_s|\to\infty}\left[\eta a_s - 2\Delta a_c - \frac{1}{4}\nu \cdot a_s(a_c^2 + a_s^2) + P\omega\right] p - \frac{1}{2\omega}\Phi_{\xi\xi}\frac{\partial p}{\partial a_s} = 0, \tag{10b}$$

The system of differential equations in Eq. (9, 10) has a closed-form solution for zero detuning, $\Delta = 0$, as discussed in [2]. This solution can be expressed using a stationary potential:

$$\begin{aligned}
p_0(a_c, a_s) &= C \exp\left(-\Psi(a_c, a_s)\right), \\
&= C \exp\left(\frac{\eta}{2S}\left(\left(a_s + \frac{P\omega}{\eta}\right)^2 + a_c^2 - \frac{\nu}{8\eta}\left(a_s^2 + a_c^2\right)^2\right)\right).
\end{aligned} \tag{11}$$

The normalization factor C has to be determined numerically for particular parameters settings.

2.1.2 Small but positive detuning considered, $\Delta > 0$

Within the lock-in interval, the stationary solution to the reduced Fokker-Planck equation for partial amplitudes can be searched for in the form of the Galerkin approximation, employing the known solution for the case of zero detuning:

$$p(a_c, a_s) = p_0(a_c, a_s)\sum_{k,l=0}^{M,k} q_{kl} \cdot a_c^{k-l} \cdot a_s^l\,. \tag{12}$$

where M is the upper limit of stochastic moments included into the analysis.

The coefficients $q_{k,l}$ for $k, l = 0, \ldots, M; k + l \leq M$ follow from the linear system

$$
\begin{aligned}
&\iint\limits_{-\infty}^{\infty} a_c^{r-s} a_s^s \frac{\partial}{\partial a_c}\left(\left(\eta a_c + 2\Delta a_s - \frac{1}{4}\nu a_c \alpha^2\right) p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) \mathrm{d}a_c \mathrm{d}a_s \\
&+\iint\limits_{-\infty}^{\infty} a_c^{r-s} a_s^s \frac{\partial}{\partial a_s}\left(\left(\eta a_s - 2\Delta a_c - \frac{1}{4}\nu a_s \alpha^2 + P\omega\right) p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) \mathrm{d}a_c \mathrm{d}a_s \\
&-\iint\limits_{-\infty}^{\infty} a_c^{r-s} a_s^s S\left[\frac{\partial^2}{\partial a_c^2} + \frac{\partial^2}{\partial a_s^2}\right]\left(p_0(a_c, a_s) \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) \mathrm{d}a_c \mathrm{d}a_s \;=\; 0,
\end{aligned}
\tag{13}
$$

where $S = \dfrac{1}{2\omega}\Phi_{\xi\xi}(\omega)$, $\alpha^2 = (a_c^2 + a_s^2)$ and $r, s = 0, \ldots, M; r + s \leq M$.

The convergence of the improper integrals in Eq. (13) is ensured by the properties of the function $p_0(a_c, a_s)$. In the degenerate case where $s = 0$ and $r = 0$cen be replaced by the normalizing condition $q_{0,0} = 1$.

The series in Eq. (12) consists of polynomial basis functions that meet the prescribed boundary conditions solely through multiplication by the exponentials in p_0. Despite the poor numerical properties of polynomial basis functions, usually resulting in an ill-conditioned Gram matrix, [21], the construction of the system matrix remains feasible for small values of M, provided careful attention is given to numerical integration handling. Several general recommendations are given in [22].

2.2 Non-stationary case

In general, when details within quasi-periods should be kept, the dependence on the original time coordinate must be retained. We start from the SDE, Eqs (1b), and write out the FPE in a form adequate to the problem discussed. Indeed, the input and output processes can be considered as Markovian, and therefore the response PDF of the system Eqs (1b) can be searched by means of the FPE.

Assuming small but sufficient detuning $\Delta \sim \varepsilon$ so that $\Delta_l < \Delta = |\omega_0 - \omega_s| < \Delta_u$, and considering terms like $(\eta - \nu u^2) \cdot \dot{u}$ and $P\omega^2$ as of small order ε, along with $h \cdot \xi(t)$ being of order $\varepsilon^{1/2}$, the FPE for the stochastic differential equations given by Eq. (1b) can be constructed in the form:

$$
\frac{\partial p(\mathbf{x}, t)}{\partial t} = -\frac{\partial}{\partial x_j}\left(\kappa_j(\mathbf{x}, t) \cdot p(\mathbf{x}, t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_k}\left(\kappa_{jk}(\mathbf{x}, t) \cdot p(\mathbf{x}, t)\right) \tag{14}
$$

where $\mathbf{x} = \{u, v\}$ with $j, k = u, v$. The parameters $\kappa_j(\mathbf{x}, t)$ and $\kappa_{jk}(\mathbf{x}, t)$ represent the first and second derivative moments, known as drift and diffusion coefficients. The input random process $\xi(t)$ is correlation stationary and ergodic, allowing the relevant stochastic moments to depend only on the difference $\tau = t_1 - t_2$. Consequently, parameters $\kappa_j(\mathbf{x}, t)$ and $\kappa_{jk}(\mathbf{x}, t)$ can be expressed as:

$$\begin{array}{rcl}\kappa_j(\mathbf{x}_t,t) & = & f_j(\mathbf{x}_t,t) \\ \kappa_{jk}(\mathbf{x}_t,t) & = & g_{jr}(\mathbf{x}_t,t)\int\limits_{-\infty}^{\infty} g_{kr}(\mathbf{x}_{t+\tau},t+\tau)R_{rr}(\tau)\mathrm{d}\tau\end{array} \tag{15}$$

where $R_{rr}(\tau)$ is the auto-correlation function of the $w_r(t)$ input process, $R_{rr}(\tau) = R(\tau)$. In this particular case,

$$\begin{array}{ll}\kappa_u = v, & \kappa_v = (\eta-\nu u^2)\cdot v - \omega_0^2 u - P\omega^2\cos\omega t, \\ g_{uu} = g_{uv} = g_{vu} = 0, & g_{vv} = h, \\ \kappa_{uu} = \kappa_{uv} = \kappa_{vu} = 0, & \kappa_{vv} = g_{vv}\displaystyle\int_{-\infty}^{\infty} g_{vv}R_{vv}(\tau)\mathrm{d}\tau = h^2\sigma_{\xi\xi}^2 = h^2\cdot S,\end{array} \tag{16}$$

where S[s] is the variance of the process $\xi(t)$ and the FPE can be readily written out as follows:

$$\begin{aligned}\frac{\partial p(\mathbf{x},t)}{\partial t} &= -\frac{\partial}{\partial u}\left(vp(\mathbf{x},t)\right) \\ &\quad -\frac{\partial}{\partial v}\left(\left((\eta-\nu u^2)v - \omega_0^2 u - P\omega^2\cos\omega t\right)p(\mathbf{x},t)\right) + \frac{1}{2}h^2 S\frac{\partial^2 p(\mathbf{x},t)}{\partial v^2},\end{aligned} \tag{17}$$

together with initial and boundary conditions:

$$\text{a)}\ \lim_{u,v\to\pm\infty} p(u,v,t) = 0 \qquad \text{b)}\ p(u,v,0) = \delta(u,v)\,. \tag{18}$$

The solution procedure for Eq. (17) along with conditions in Eqs (18) can be executed using the Galerkin series, formulating the solution as:

$$p(u,v,t) = p_0(u,v)\sum_{kl=0}^{M,k} q_{kl}(u,v,t) \tag{19}$$

The series in Eq. (19) represents an approach to a weak solution to the FPE equation in the probabilistic mean of the term. The elements q_{kl} are composed similarly to those in Eq. (12), formulated as:

$$q_{kl}(u,v,t) = q_{kl}(t)L_{k-l}(\alpha u)L_l(\beta v), \qquad \alpha^2 = \frac{\omega_b\omega_0^2}{h^2 S}, \quad \beta^2 = \frac{\omega_b}{h^2 S}\,, \tag{20}$$

where $L_{r-s}(\alpha u)$ or $H_s(\beta v)$ are l'Hermite polynomials. Polynomial orders are introduced such that within one value of $k = 1,..,M$, the polynomials of equivalent order k are addressed. The function $p_0(u,v)$ serves as a weight during the construction of the relevant differential system. It can be adopted in the form of Boltzmann's solution to a related problem without damping and external excitation. For details, refer to [23–25], and other relevant papers and monographs, particularly:

$$p_0(u,v) = D\cdot\exp(-\frac{2\omega_b}{h^2 S}H(u,v)) \tag{21}$$

where D is the dimensionless normalizing constant, which can be put for now $D = 1$. $H(u, v)$ represents the Hamiltonian function of the basic system. In particular:

$$H(u, v) = \frac{1}{2}\omega_0^2 u^2 + \frac{1}{2}v^2 \tag{22}$$

which implicates:

$$p_0(u, v) = p_u(u) \cdot p_v(v) \quad \Rightarrow \quad p_u(u) = \exp(-\alpha u^2), \quad p_v(v) = \exp(-\beta v^2), \tag{23}$$

indicating that u, v are stochastically independent Gaussian processes on a level of the zero-th approximation.

There are several options for approximation the functions $q_j(u, v, t)$ in Eq. (19). The generalized method of stochastic moments, [7], can be based on the selection of l'Hermite polynomials, orthogonal with the weight given in Eqs (21); enables simplification in subsequent steps. The generalization involves the introduction of moments with polynomials, not solely relying on factors like $u^{k-l}v^l$. However, the physical interpretation may become somewhat more intricate compared to using the series in Eq. (12).

The weak formulation with respect to Eq. (17) can be obtained using product of l'Hermite polynomials in variables u, v as the test functions. After integration over the whole space and several per partes operations, and respecting the boundary conditions with respect to $u, v \to \pm\infty$, one obtains:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{E}\{\Phi(u, v)\} = h^2 S\,\mathbf{E}\left\{\frac{\partial^2\Phi(u, v)}{\partial v^2}\right\} + \mathbf{E}\left\{\frac{\partial\Phi(u, v)}{\partial u}v\right\} + \\ \mathbf{E}\left\{\frac{\partial\Phi(u, v)}{\partial v}\left(\eta v - \nu u^2 v - \omega_0^2 u - P\omega^2\cos\omega t\right)\right\} \end{aligned} \tag{24}$$

where $\mathbf{E}\{\cdot\}$ represents the operator of mathematical mean value with respect to unknown PDF $p(u, v, t)$. The series (19) contains unknowns $q_{kl}(t)$ and initial PDF $p_0(u, v)$. Therefore it is convenient to express Eq. (24) with reference to the initial PDF $p_0(u, v)$ given by Eq. (21). Taking into account Eq. (19), following general relation between mathematical mean value of a function $\Phi(u, v)$ with respect to $p(u, v, t)$ and $p_0(u, v)$ can be established:

$$\begin{aligned} \mathbf{E}\{\Phi(u, v)\} &= \iint_{-\infty}^{\infty} \Phi(u, v) p(u, v, t)\mathrm{d}u\mathrm{d}v \\ &= \iint_{-\infty}^{\infty} \Phi(u, v) p_0(u, v) \sum_{kl=0}^{M,k} q_{kl}(u, v, t)\mathrm{d}u\mathrm{d}v \\ &= \sum_{kl=0}^{M,k} \mathbf{E}_0\{\Phi(u, v) q_{kl}(u, v, t)\} \end{aligned} \tag{25}$$

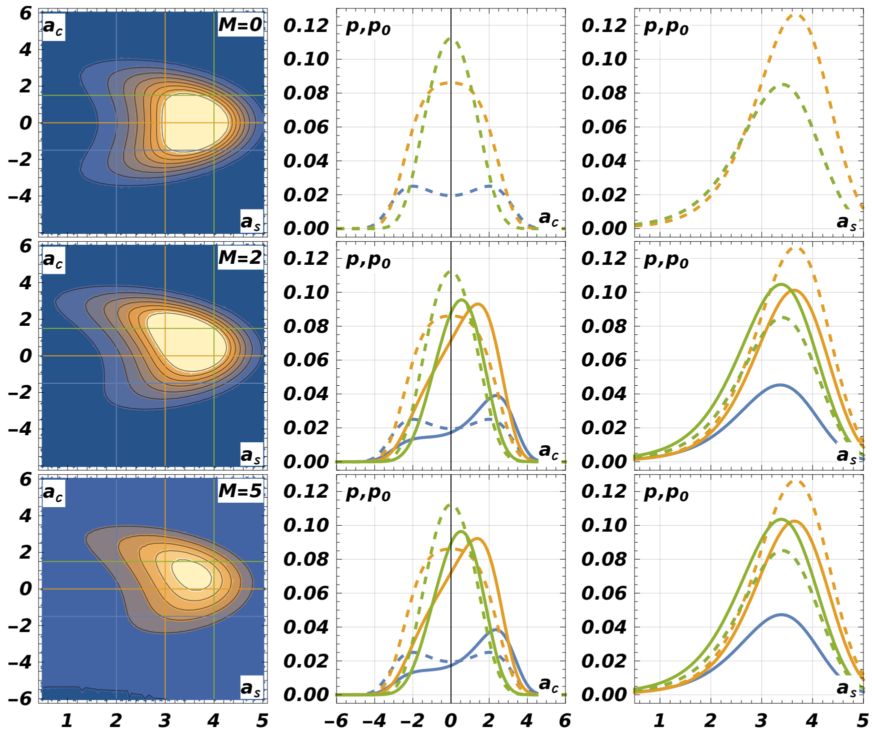

Figure 1: The Galerkin approximation of the stationary cross-PDF for increasing number of stochastic moments and detuning value $\Delta = 0.10$

The particular choice of the l'Hermite polynomial series of increasing degree enables the formulation of a system of ordinary differential equations approximating the time-dependent term $q_{kl}(t)$ in Eq. (20). However, this formulation has not yet been completed.

3 Numerical example

The response of the van der Pol oscillator in Eqs (1a) or (1b) is stationary in the lock-in region, which, for the natural frequency of the van der Pol system at $\omega_0 = 1$, corresponds to interval $\omega \in (0.85, 1.35)$. In terms of detuning, the lock-in interval corresponds to $\Delta \in (-0.09, 0.13)$. Other values used in the numerical example are $\eta = 1/2$, $\nu = 1/4$, and $P = 1$.

The PDF of the stationary response, computed using the proposed procedure, is illustrated in Fig. 1. PDFs of both partial amplitudes a_c, a_s are shown for three values $M = 0, 2, 5$ and a non-negligible value of detuning $\delta = 0.05$. In each row, i.e., for each choice of M, the contour plot of the estimated cross-PDF $p(a_c, a_s)$ is shown on the left. The middle plot depicts the sections of the PDF for fixed values $a_c = \{-3/2.0, 3/2\}$ and the right-hand plot illustrates the sections for the selected values $a_s = \{2, 3, 4\}$. The sections and the corresponding colors are indicated as horizontal/vertical lines in the left-hand plots. The stochastic parameters used in all examples are $h = 1$ and $S = 1$.

The first row for $M = 0$ shows the analytical solution for which no Galerkin correction is applied; the sections are plotted in dashed curves and are repeated in all remaining graphs as the reference values. Note, for example, that in the left column the blue and green dashed lines coincide due to the symmetry in the a_c variable. The curves corresponding to the PDF estimates for higher values of M are shown in solid. It is apparent from the plot that the most significant contribution to the analytic solution is represented by the low value of M, while the higher order moments bring no significant changes to the PDF estimate.

4 Concluding remarks

A comprehensive analytical-numerical approach for estimating the probability density function of the solution to the stochastic differential equation has been demonstrated using the single-degree-of-freedom van der Pol equation subjected to combined harmonic and random excitation as an illustrative example. Three distinct response and solution regimes have been identified: stationary with no detuning, allowing for an explicit solution based on stochastic averaging of partial amplitudes; stationary with small detuning, where the solution was obtained through Galerkin approximations based on polynomial basis functions; and a non-stationary case characterized by a detuning larger than a certain limit. The procedure for the non-stationary solution was outlined, relying on a modified Galerkin method with orthogonal polynomials as basis and test functions, followed by solving the Ordinary Differential Equation (ODE) system to obtain the time-dependent stochastic moments of the response. This non-stationary solution procedure is an ongoing area of research.

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References

[1] P. Lou, J. D. Yau, F. T. K. Au, S. Urushadze, “Shifted resonance of railway bridges under trains passing by each other”, International Journal of Structural Stability and Dynamics, 22, 2022, https://doi.org/10.1142/s0219455422710018.

[2] J. Náprstek, C. Fischer, S. Pospíšil, A. Trush, “Modeling of the quasi-periodic galloping response type under combined harmonic and random excitation”, Computers & Structures, 247:106478, 2021, https://doi.org/10.1016/j.compstruc.2020.106478.

[3] H. P. Langtangen, “A general numerical solution method for Fokker-Planck equations with applications to structural reliability”, Probabilistic Engineering Mechanics, 6:33–48, 1991, https://doi.org/10.1016/s0266-8920(05)80005-0.

[4] L. Pichler, A. Masud, L. A. Bergman, “Numerical Solution of the Fokker–Planck Equation by Finite Difference and Finite Element Methods—A Comparative Study”, “Computational Methods in Stochastic Dynamics”, Springer Netherlands, 69–85, 2013, https://doi.org/10.1007/978-94-007-5134-7_5.

[5] C. S. Manohar, “Methods of nonlinear random vibration analysis”, Sadhana, 20:345–371, 1995, https://doi.org/10.1007/bf02823196.

[6] S. H. Crandall, “Perturbation techniques for random vibration of nonlinear systems”, J. Acoust. Soc. Am., 35:1700–1705, 1963, https://doi.org/10.1121/1.1918792.

[7] V. S. Pugachev, I. N. Sinitsyn, Stochastic differential systems — Analysis and filtering, J. Willey, Chichester, 1987.

[8] Z. L. Huang, Z. H. U. W.q., Y. Suzuki, “Stochastic averaging of strongly non-linear oscillators under combined harmonic and white-noise excitations”, Journal of Sound and Vibration, 238:233–256, 2000, https://doi.org/10.1006/jsvi.2000.3083.

[9] R. N. Iyengar, P. K. Dash, “Study of the Random Vibration of Nonlinear Systems by the Gaussian Closure Technique”, Journal of Applied Mechanics, 45:393–399, 1978, https://doi.org/10.1115/1.3424308.

[10] S. H. Crandall, “Non-gaussian closure for random vibration of non-linear oscillators”, International Journal of Non-Linear Mechanics, 15:303–313, 1980, https://doi.org/10.1016/0020-7462(80)90015-3.

[11] Q. Liu, H. Davies, “The non-stationary response probability density functions of non-linearly damped oscillators subjected to white noise excitations”, Journal of Sound and Vibration, 139:425–435, 1990, https://doi.org/10.1016/0022-460x(90)90674-o.

[12] R. C. Booton, “Nonlinear control systems with random inputs”, IRE Transactions on Circuit Theory, 1:9–18, 1954, https://doi.org/10.1109/tct.1954.6373354.

[13] K. Wang, Z. Zhu, L. Xu, “Transient probabilistic solutions of stochastic oscillator with even nonlinearities by exponential polynomial closure method”, Journal of Vibration and Control, 28:1086–1094, 2021, https://doi.org/10.1177/1077546320987778.

[14] J. Luo, G.-K. Er, V. P. Iu, C. C. Lam, “Transient probabilistic solution of stochastic oscillator under combined harmonic and modulated Gaussian white noise stimulations”, Nonlinear Dynamics, 111:17709–17723, 2023, https://doi.org/10.1007/s11071-023-08810-2.

[15] G.-P. Bai, G.-K. Er, V. P. Iu, C. C. Lam, “Optimization-oriented exponential-polynomial-closure approach for analyzing nonlinear stochastic oscillators”, Probabilistic Engineering Mechanics, 73:103477, 2023, https://doi.org/10.1016/j.probengmech.2023.103477.

[16] Y. K. Lin, G. Q. Cai, Probabilistic structural dynamics, McGraw-Hill, New York, 1995.

[17] J. Náprstek, C. Fischer, “Trajectories of a ball moving inside a spherical cavity using first integrals of the governing nonlinear system”, Nonlinear Dynamics, 106:1591–1625, 2021, https://doi.org/10.1007/s11071-021-06709-4.

[18] R. L. Stratonovich, Topics in the theory of random noise, Gordon and Breach, New York, 1963.

[19] R. Z. Khasminskii, Stochastic Stability of Differential Equations, number 7 in Mechanics: Analysis, Sijthoff & Noordhoff International Publishers B.V., Alphen aan den Rijn, The Netherlands, 1 edition, 1980.

[20] R. Khasminskii, Stochastic stability of differential equations, Springer-Verlag GmbH, Berlin, 2012.

[21] J. M. Taylor, “The condition of gram matrices and related problems”, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 80:45–56, 1978, https://doi.org/10.1017/s030821050001012x.

[22] J. Náprstek, C. Fischer, “Averaging-based characteristics of the response induced by combined random and harmonic excitation.”, P. Sassi, S.and Biswas, J. Náprstek, eds., “Proceedings of the 15th International Conference on Vibration Problems. ICoVP 2023”, Springer, Singapore, 2024, Lecture Notes in Mechanical Engineering., https://doi.org/10.1007/978-981-99-5922-8_18, in print.

[23] J. Náprstek, “Application of the maximum entropy principle to the analysis of non-stationary response of SDOF/MDOF systems”, L. Půst, F. Peterka, eds., “Proc. 2nd European Non-Linear Oscillations Conference — EUROMECH”, IT ASCR, Prague, 305–308, 1996.

[24] G. Q. Cai, Y. K. Lin, “On exact stationary solutions of equivalent non-linear stochastic systems”, International Journal of Non-Linear Mechanics, 23:315–325, 1988.

[25] W. Q. Zhu, G. Q. Cai, Y. K. Lin, “On exact stationary solutions of stochastically perturbed Hamiltonian systems”, Probabilistic Engineering Mechanics, 5:84–87, 1990.