Proceedings of the Fifteenth International Conference on
Computational Structures Technology
Edited by: P. Iványi, J. Kruis and B.H.V. Topping
Civil-Comp Conferences, Volume 9, Paper 8.3
Civil-Comp Press, Edinburgh, United Kingdom, 2024
ISSN: 2753-3239, doi: 10.4203/ccc.9.8.3

Lévy's Solution for Laminated Composite Plates using Higher-Order Shear and Normal Deformation Theory

H. Sawhney[1], S. Yadav[2], Y. Desai[1] and S. Pendhari[2]

[1]Department of Civil Engineering, Indian Institute of Technology Bombay, Mumbai, India
[2]Department of Structural Engineering, Veermata Jijabai Technological Institute, Mumbai, India

Abstract

Equivalent single layer (ESL) theories have been extensively used in the analysis of plates. One of the common assumptions that are considered in all the ESL theories is that the thickness is small as compared to the in-plane dimensions. This assumption is the basis for converting a 3D plate problem into a 2D plate problem by assuming a displacement-model along the thickness. Higher-order shear deformation theories (hosts) consider more realistic non-linear variation of displacements along the thickness as compared to the other ESL theories i.e., the classical plate theory (CPT) and first-order shear deformation theories (FOSTS), which consider the linear variation. Due to this reason the solutions obtained using hosts are closer to the elasticity solutions. In this paper, static solution of the laminated composite plates is provided using 12 degrees of freedom higher-order shear and normal deformation theory (HOSNT). Results are obtained using the state-space approach for Lévy-type plates i.e., two opposite plate edges having simply-supported boundary condition and other two plate edges having combination of simply-supported, clamped and free boundary conditions. Results obtained compare well with the corresponding results available in the literature.

Keywords: higher-order shear and normal deformation theory, Levy's solution, laminated plates, state-space technique, numerical solution, semi-analytical.

1 Introduction

Composite materials are increasingly replacing conventional materials due to their excellent engineering properties. However, they exhibit a complex structural response, with local failures originating along weak planes and progressing until structural failure occurs. One such weak plane exists at the interface between adjacent layers in layered components, where delamination can initiate due to high inter-laminar shear and normal stresses. Therefore, it is crucial to study and evaluate inter-laminar stresses for accurate prediction of potential delamination. Various researchers globally have proposed several plate theories to investigate the behavior of composite materials. These theories can be broadly classified into ESL, zig-zag, and layer-wise theories. ESL theories aim to simplify the complex 3D problem into a more feasible 2D problem by making specific assumptions. Among the notable ESL theories are the CPT, FOSTS and higher-order shear deformation theories (hosts). These theories are extensively employed to analyze the response of composite materials and play a vital role in comprehending their characteristics.
The 3D elasticity solutions are termed as the exact solutions, and they are used as the reference to validate the corresponding numerical results from different other techniques. Srinivas et al. [1,2] developed a linear, three-dimensional elasticity and small deformation exact solution technique for thick rectangular plates. Pagano [3–5] developed an elasticity solution for cross-ply laminated and sandwich plates. The solution is presented for cylindrical bending of long rectangular plate and finite rectangular plate subjected to bi-directional sinusoidal loading. The simpler ESL theories are required due to the limitations of elasticity theories being able to solve for specific cases of material and boundary conditions. The CPT disregards transverse shear effects, leading to underestimation of displacements and overestimation of buckling loads in moderately thick and thick plates. The FOSTS introduced by Reissner [6] and Mindlin [7] address this drawback by considering shear deformation effects, providing more accurate results for moderately thick plates. However, the limitations of the CPT and the fost restricted it to calculate the crucial transverse shear stress values. Hence, another class of ESL theories known as hosts were developed. They assume a more realistic non-linear variation of displacements across the thickness. Kant [8] had proposed a refined higher-order theory having 6 degrees of freedom for symmetric plates. A remarkably popular third-order shear deformation theory was developed by Reddy [9] and parallelly by Levinson [10] and Murthy [11]. Kant and Swaminathan [12]–[14] performed extensive study using higher-order theories with different degrees of freedom using Navier’s approach.
Flexure analysis of the laminated composite plates using 12 degrees of freedom HOSNT, has been the primary aim of the present work. The solutions for Lévy-type plates using HOSNT will be attempted first time and hence will be the novelty of the paper.

2 Problem Formulation

A laminated plate consisting of a number of isotropic/orthotropic lamina is considered. The length, width, and thickness of the rectangular plate are considered as a, b and h, respectively, along with a right-handed cartesian coordinate system $(x - y - z)$ as shown in Fig. 1. Both the lamina and the laminate are assumed to have a constant thickness. The fibre orientation angle θ is measured in anti-clockwise direction from x-axis.

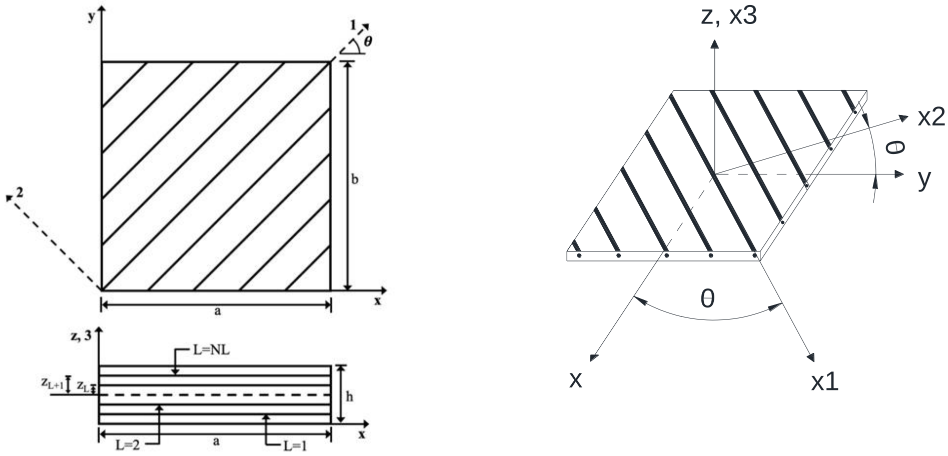

Figure 1: Geometry of plate, positive reference axes for laminate and fibre orientation.

The displacement field for the HOSNT that is used in the present analysis is defined in Eq. (1).

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y) \\
v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y) \\
w(x,y,z) &= w_0(x,y) + z\theta_z(x,y) + z^2 w_0^*(x,y) + z^3\theta_z^*(x,y)
\end{aligned}
\quad (1)
$$

Using the linear strain-displacement and three-dimensional constitutive relationship, the governing differential equations (gde) are derived by applying the minimization potential energy principle i.e., the variation of the total potential energy of the plate must be zero ($\delta\Pi = 0$). The 12 governing differential equations obtained are given in Eq. (2).

$$
\begin{aligned}
&\delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 && \delta u_0^* : \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = 0 \\
&\delta v_0 : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 && \delta v_0^* : \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = 0 \\
&\delta w_0 : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z^t = 0 && \delta w_0^* : \frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z^* + \frac{h^2}{4}\left(q_z^t\right) = 0 \\
&\delta \theta_x : \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 && \delta \theta_x^* : \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = 0 \\
&\delta \theta_y : \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 && \delta \theta_y^* : \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = 0 \\
&\delta \theta_z : \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + \frac{h}{2}\left(q_z^t\right) = 0 && \delta \theta_z^* : \frac{\partial S_x^*}{\partial x} + \frac{\partial S_y^*}{\partial y} - 3N_z^* + \frac{h^3}{8}\left(q_z^t\right) = 0
\end{aligned}
\tag{2}
$$

Simultaneously, along with the gde the derivation also produces the natural boundary conditions, which are mentioned as below. It is necessary to prescribe one of each of the following 24 products along the edges of the plate to meet the boundary conditions.

$$u_0 N_{xy}, u_0^* N_{xy}^*, v_0 N_y, v_0^* N_y^*, w_0\, Q_y, w_0^* Q_y^*, \theta_x M_{xy}, \theta_x^* M_{xy}^*, \theta_y M_y, \theta_y^* M_y^*, \theta_z S_y, \theta_z^* S_y^* \quad \text{for } x = 0 \text{ and } x = a$$

$$u_0 N_x, u_0^* N_x^*, v_0 N_{xy}, v_0^* N_{xy}^*, w_0\, Q_x, w_0^* Q_x^*, \theta_x M_x, \theta_x^* M_x^*, \theta_y M_{xy}, \theta_y^* M_{xy}^*, \theta_z S_x, \theta_z^* S_x^* \quad \text{for } y = 0 \text{ and } y = b$$

Taking cue from the above BCs, the commonly occurring boundary conditions for different cases of Lévy-type plates are given as follows,

The Simply-Supported (S) boundary conditions at edges $y = 0$ and $y = b$ are,
$u_0 = u_0^* = w_0 = w_0^* = \theta_x = \theta_x^* = \theta_z = \theta_z^* = M_y = M_y^* = N_y = N_y^* = 0$.

While the boundary conditions for the remaining two edges $x = 0$ and $x = a$ will be,

Simply-Supported (S):

$v_0 = v_0^* = w_0 = w_0^* = \theta_y = \theta_y^* = \theta_z = \theta_z^* = M_x = M_x^* = N_x = N_x^* = 0$.

Clamped (C): $u_0 = u_0^* = v_0 = v_0^* = w_0 = w_0^* = \theta_x = \theta_x^* = \theta_y = \theta_y^* = \theta_z = \theta_z^* = 0$.

Free (F): $N_x = N_x^* = N_{xy} = N_{xy}^* = M_x = M_x^* = M_{xy} = M_{xy}^* = Q_x = Q_x^* = S_x = S_x^* = 0$.

Using this kinematics of HOSNT the solution procedure is developed to solve the partial gde.

3 Solution Procedure

For the flexural analysis of orthotropic and cross-ply laminated, the Lévy-type solution is employed in conjunction with the state-space technique. The procedure starts with assuming the primary unknowns in the form of single-term trigonometric Fourier series. This reduces the two-dimensional problem into a one-dimensional boundary value problem (bvp) in x-coordinate. The resulting bvp as shown in Eq. (3) is then solved using the state-space approach [15].

$$\left\{\frac{dZ}{dx}\right\}_{24\times 1} = \left\{Z'\right\}_{24\times 1} = \left[T\right]_{24\times 24}\left\{Z\right\}_{24\times 1} + \left\{F\right\}_{24\times 1} \tag{3}$$

The solution for the bvp in Eq. (3) can be obtained using the following expression.

$$\left\{Z(x)\right\} = [V]\begin{bmatrix} e^{D_1 x} & & 0 \\ & \ddots & \\ 0 & & e^{D_{24} x} \end{bmatrix}[V]^{-1}\left\{Z_0\right\} + \int_{\eta=0}^{\eta=x} e^{[T](x-\eta)}\left\{F\right\}\mathrm{d}\eta \tag{4}$$

where, D_i ($i = 1$ to 24) are the distinct eigenvalues of $[T]$ and, $[V]$ and $[V]^{-1}$ are the matrices of corresponding eigenvectors and their inverse, respectively. The constant vector $\{P\} = [V]^{-1}\{Z_0\}$ in Eq. (4) is calculated using the boundary conditions at $x = 0$ and $x = a$. Note that it is crucial to use vector $\{P\}$ (Eq. (4)) as the matrix $[V]^{-1}$ is an ill-conditioned matrix and hence give erroneous solutions if left untreated.

Using the solution procedure, the numerical results for various cases are presented in the next section.

4 Numerical Results and Discussions

The computer codes incorporating the methodology explained in previous section is developed in the MATLAB coding language. Numerical examples pertaining to the flexure analysis of the laminated cross-ply plates using HOSNT are presented in this section.

To prove the efficacy of the present theory and methodology the square three-layered symmetric cross-ply laminated (0/90/0) plate with all the edges having simply-supported boundary condition (SSSS) is analysed. The results are compared with the corresponding results available from different literatures.

The material property used for the analysis and the normalization scheme adopted for the result comparison are given in Table 1 and Eq. (5), respectively.

$$\bar{w} = w\left(\frac{100E_2h^2}{q_0a^4}\right);\ \left\langle \bar{\sigma}_x \quad \bar{\sigma}_y \quad \bar{\tau}_{xy}\right\rangle = \left\langle \sigma_x \quad \sigma_y \quad \tau_{xy}\right\rangle\left(\frac{h^2}{q_0a^2}\right); \tag{5}$$

$$\langle \bar{\tau}_{xz} \quad \bar{\tau}_{yz} \rangle = \langle \tau_{xz} \quad \tau_{yz} \rangle \left(\frac{h}{q_0 a} \right)$$

Table 1: Material Properties

Source	Elastic constants		
Pagano [3]	$E_1 = 25E_2$ $E_2 = E_3 = 10^6$ psi	$\nu_{12} = 0.25$ $\nu_{13} = 0.25$ $\nu_{23} = 0.25$	$G_{12} = G_{13} = 0.50E_2$ $G_{23} = 0.20E_2$

It can be seen from Table 2 and Table 3 that the results from the present analysis are in good agreement with the corresponding results from elasticity solutions. Comparing other ESL theories with the elasticity solutions shows that while HSDT-R is giving considerable good results, the other two theories, i.e., HSDT-S and FSDT have relatively significant errors. It can be confidently concluded that the results obtained from HOSNT are not only in good agreement but also somewhat more in proximity with the exact elasticity solutions compared to the results from other ESL theories.

Table 2: Normalized displacement and in-plane stresses ($\bar{w}, \bar{\sigma}_x, \bar{\sigma}_y$) for different length-to-thickness ratio (a/h) of a square three-layered cross-ply (0/90/0) laminated SSSS plate.

a/h	*Source*	$\bar{w}\left(\frac{a}{2},\frac{b}{2};0\right)$	$\bar{\sigma}_x\left(\frac{a}{2},\frac{b}{2};\frac{h}{2}\right)$	$\bar{\sigma}_y\left(\frac{a}{2},\frac{b}{2};\frac{h}{6}\right)$
10	[Ω]Elasticity Sol.[SL]	-	-0.5900	-0.2880
	Present[SL]	0.7151	-0.5832	-0.2732
	%Error	-	*-1.15%*	*-5.14%*
	[R]HSDT-R	0.7125	-0.5684	-0.2690
	%Error	-	*-3.66%*	*-6.60%*
	[S]HSDT-S	0.6041	-0.5747	-0.1649
	%Error	-	*-2.59%*	*-42.74%*
	[M]FSDT	0.6693	-0.5134	-0.2536
	%Error	-	*-12.98%*	*-11.94%*
50	[Ω]Elasticity Sol.[SL]	-	-0.5410	-0.1850
	Present[SL]	0.4432	-0.5406	-0.1838
	%Error	-	*-0.07%*	*-0.65%*
	[R]HSDT-R	0.4430	-0.5399	-0.1836
	%Error	-	*-0.20%*	*-0.76%*
	[S]HSDT-S	0.4382	-0.5401	-0.1790
	%Error	-	*-0.17%*	*-3.24%*
	[M]FSDT	0.4411	-0.448	-0.1829
	%Error	-	*-17.19%*	*-1.14%*

Source Ref.: [$] Pendhari et al. [16,17]; [Ω]Pagano [3]; [R]Reddy [9,12]; [S]Senthilnathan et al. [12,18]; [M]Reddy et al. [12,19]
[SL]Bi-directional sinusoidal loading
Numbers in % are the percentage error w.r.t elasticity solutions.

Table 3: Normalized in-plane and transverse shear stresses ($\bar{\tau}_{xy}, \bar{\tau}_{xz}, \bar{\tau}_{yz}$) for different length-to-thickness ratio (a/h) of a square three-layered cross-ply (0/90/0) laminated SSSS plate.

a/h	*Source*	$\bar{\tau}_{xy}\left(0,0;\frac{h}{2}\right)$	$\bar{\tau}_{xz}\left(0,\frac{b}{2};0\right)^{E}$	$\bar{\tau}_{yz}\left(\frac{a}{2},0;0\right)^{E}$
10	$^{\Omega}$Elasticity Sol.^{SL}	-0.0290	0.3570	0.1228
	Present^{SL}	-0.0279	0.3660	0.1180
	%Error	-3.79%	2.52%	-3.91%
	^{R}HSDT-R	-0.0277	-	0.1167
	%Error	-4.48%	-	-4.97%
	^{S}HSDT-S	-0.0227	-	-
	%Error	-21.72%	-	-
	^{M}FSDT	-0.0252	-	0.1108
	%Error	-13.10%	-	-9.77%
50	$^{\Omega}$Elasticity Sol.^{SL}	-0.0216	0.3930	0.0842
	Present^{SL}	-0.0216	0.3930	0.0842
	%Error	*0.00%*	*0.00%*	*0.00%*
	^{R}HSDT-R	-0.0216	-	-
	%Error	*0.00%*	-	-
	^{S}HSDT-S	-0.0213	-	-
	%Error	*-1.39%*	-	-
	^{M}FSDT	-0.0215	-	-
	%Error	*-0.46%*	-	-

Source Ref.: $ Pendhari et al. [16,17]; $^{\Omega}$Pagano [3]; ^{R}Reddy [9,12]; ^{S}Senthilnathan et al. [12,18]; ^{M}Reddy et al. [12,19]

^{SL}Bi-directional sinusoidal loading

^{E}Results obtained through Equilibrium equations using Navier's technique (Ref. [12]).

Numbers in % are the percentage error w.r.t elasticity solutions.

Variation of normalised transverse displacement ($\bar{w}$) along the thickness of the plates are shown in Figure 2 for the cross-ply case. The cubic/non-linear variation of $\bar{w}$ due to the displacement model for present theory can very well be seen in the figure.

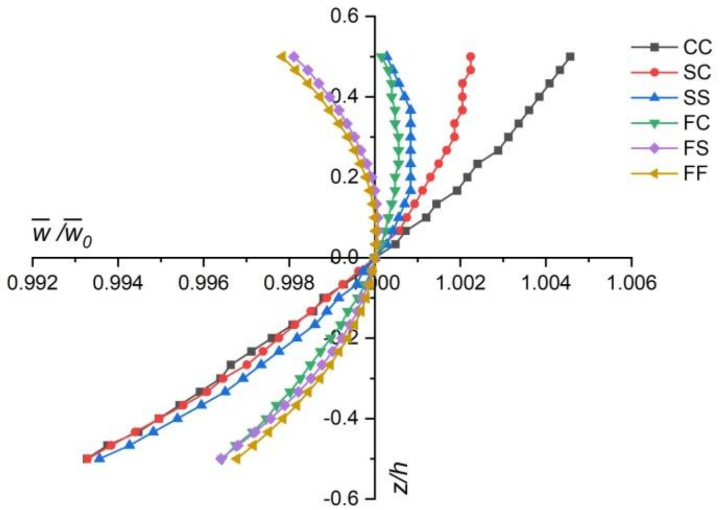

Figure 2: Variation of the ratio of normalised transverse displacement with respective midplane displacement $\overline{w}\left(\frac{a}{2},\frac{b}{2};z\right)/\overline{w}\left(\frac{a}{2},\frac{b}{2};0\right)$ for cross-ply (0/90/0) laminated plate $(a/h=10)$.

Effect of plate length-to-thickness ratio (a/h) on the values of normalised transverse displacement is presented in Figure . This reduction of values of normalised transverse displacement with the increase in the values of a/h is because of shear deformation effect which is more prominent in thick plates as compared to thin plates.

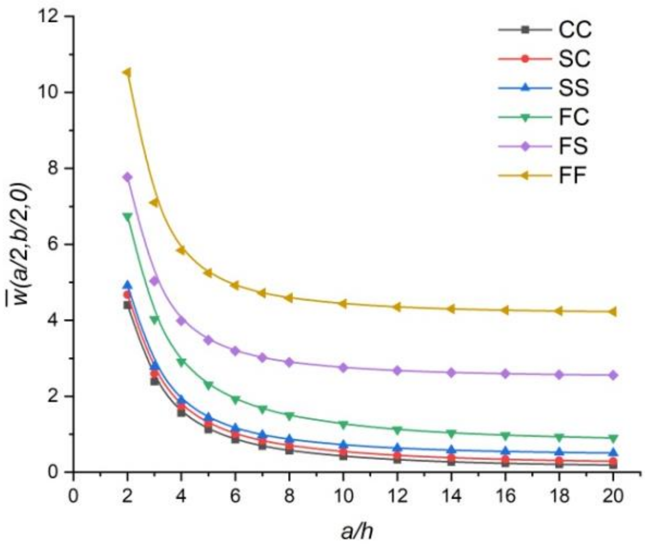

Figure 3: Effect of plate length-to-thickness ratio (a/h) on the normalised transverse displacement $(\overline{w})$ for cross-ply (0/90/0) laminated plate.

Effect of plate length-to-width ratio (a/b) on the values of normalised transverse displacement is presented in Figure 4. The reduction in the values of normalised transverse displacement and in-plane normal stresses with the increase in the values of a/b is due to the eventual increase in the values of a/h i.e., the slenderness effect.

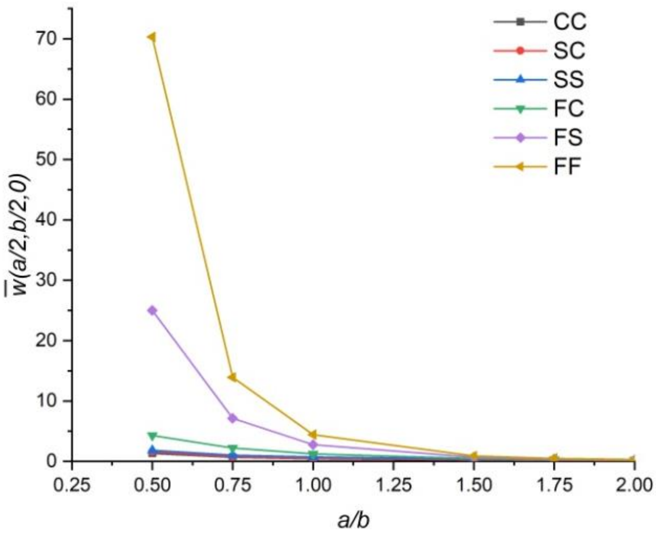

Figure 4: Effect of plate aspect ratio (a/b) on the normalised transverse displacement $(\overline{w})$ for cross-ply (0/90/0) laminated plate $(b/h = 10)$.

5. Conclusion

The study of flexure analysis using 12 DOF higher-order shear and normal deformation theory has been presented. The novelty involved using the Lévy-type plates in conjunction with the state-space numerical technique for HOSNT. The results from the present theory were in better quantitative agreement with the exact elasticity solutions as compared to the other ESL theories. The parametric study performed can serve as the benchmark solutions for the future references. It can be concluded that the higher number of primary unknowns (up to cubic) can improve the accuracy of the results with limitation of increased computational cost.

References

[1] S. Srinivas, A.K. Rao, C.V.J. Rao, "Flexure of Simply Supported Thick Homogeneous and Laminated Rectangular Plates", ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift Für Angewandte Mathematik Und Mechanik 1969; 49:449–58.

https://doi.org/10.1002/zamm.19690490802.

[2] S. Srinivas, C.V.J. Rao, A.K. Rao, "An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates", Journal of Sound and Vibration 1970;12:187–99. https://doi.org/10.1016/0022-460X(70)90089-1.

[3] N.J. Pagano, "Exact Solutions for Rectangular Bidirectional Composites and

Sandwich Plates", Journal of Composite Materials 1970;4:20–34. https://doi.org/10.1177/002199837000400102.

[4] N. J. Pagano, "Exact Solutions for Composite Laminates in Cylindrical Bending", Journal of Composite Materials 1969;3:398–411. https://doi.org/10.1177/002199836900300304.

[5] N.J. Pagano, "Influence of Shear Coupling in Cylindrical. Bending of Anisotropic Laminates", Journal of Composite Materials 1970;4:330–43.https://doi.org/10.1177/002199837000400305.

[6] E. Reissner, "The Effect of Transverse Shear Deformation on the Bending of Elastic Plates", J Appl Mech 1945:A69–77.

[7] R.D. Mindlin, " Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates", The Collected Papers of Raymond D Mindlin Volume I 1989;18:225–32. https://doi.org/10.1007/978-1-4613-8865-4_29.

[8] T. Kant, "Numerical Analysis of Thick Plates", Computer Methods in Applied Mechanics and Engineering 1982;31:1–18. https://doi.org/10.1016/0045-7825(82)90043-3.

[9] J. N. Reddy, "A Simple Higher-Order Theory for Laminated Composite Plates", Journal of Applied Mechanics 1984;51:745–52. https://doi.org/10.1115/1.3167719.

[10] M. Levinson M, "An Accurate, Simple Theory of the Statics and Dynamics of Elastic Plates", Mechanics Research Communications 1980;7:343–50.https://doi.org/10.1016/0093-6413(80)90049-X.

[11] M.V.V. Murthy, "An Improved Transverse Shear Deformation Theory for Laminated Antisotropic Plates", 1981.

[12] T. Kant, K. Swaminathan, "Analytical Solutions for the Static Analysis of Laminated Composite and Sandwich Plates based on a Higher Order Refined Theory", Composite Structures 2002;56:329–44. https://doi.org/10.1016/S0263-8223(02)00017-X.

[13] Kant T, Swaminathan K. Analytical solutions for Free Vibration of Laminated Composite and Sandwich Plates based on a Higher Order Refined Theory", Composite Structures 2001;53:73–85. https://doi.org/10.1016/S0263-8223(00)00180-X.

[14] T. Kant, K. Swaminathan, "Analytical Solutions using a Higher Order Refined Theory for the Stability Analysis of Laminated Composite and Sandwich Plates", Structural Engineering & Mechanics 2000;10:337–57.

[15] H. T. Thai, S. E. Kim, "Analytical Solution of a Two Variable Refined Plate Theory for Bending Analysis of Orthotropic Levy-type Plates", International Journal of Mechanical Sciences 2012;54:269–76. https://doi.org/10.1016/j.ijmecsci.2011.11.007.

[16] S. S. Pendhari, "A New Partial Discretization Technique in Elastostatics

with Special Reference to Laminated Composites and Sandwiches", Doctoral dissertation. Indian Institute of Technology Bombay, 2006.

[17] T. Kant, A. B. Gupta, S. S. Pendhari, Y. M. Desai, "Elasticity Solution for Cross-ply Composite and Sandwich Laminates", Composite Structures 2008;83:13–24. https://doi.org/10.1016/j.compstruct.2007.03.003.

[18] N. R. Senthilnathan, S. P. Lim, K. H. Lee, S. T. Chow, "Buckling of Shear-Deformable Plates", AIAA Journal 1987;25:1268–71. https://doi.org/10.2514/3.48742.

[19] J. N. Reddy, W. C. Chao, " A Comparison of Closed-form and Finite-element Solutions of Thick Laminated Anisotropic Rectangular Plates", Nuclear Engineering and Design 1981;64:153–67. https://doi.org/10.1016/0029-5493(81)90001-7.